Lesson 3 - Matter-Radiation Interaction
Unit 3.1 Minimal coupling and dipole approximation

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Structure of Matter - MSc in Physics
Minimal coupling (1)

Let us consider the hydrogen atom with Hamiltonian

\[
\hat{H}_{\text{matt}} = \frac{\hat{p}^2}{2m} - \frac{e^2}{4\pi\epsilon_0 |r|},
\]

(1)

where \(\hat{p} = -i\hbar \nabla\) is the linear momentum operator of the electron in the state \(|p\rangle\), and \(e > 0\) is the modulus of the electric charge of the electron. The minimal coupling with the electromagnetic field is obtained with the substitution

\[
\hat{p} \rightarrow \hat{p} + e\hat{A}(r, t)
\]

(2)

where \(\hat{A}(r, t)\) is the vector potential of the electromagnetic field. In this way we have

\[
\hat{H}_{\text{matt, shift}} = \frac{\left(\hat{p} + e\hat{A}(r, t)\right)^2}{2m} - \frac{e^2}{4\pi\epsilon_0 |r|}
\]

\[
= \hat{H}_{\text{matt}} + \frac{e}{m} \hat{A}(r, t) \cdot \hat{p} + \frac{e^2}{2m} \hat{A}(r, t)^2.
\]

(3)
Dipole approximation (I)

The dipole approximation means

\[ \hat{H}_{\text{matt, shift}} \approx \hat{H}_{\text{matt}} + \hat{H}_D, \]  

(4)

where

\[ H_D = \frac{e}{m} \hat{A}(0,0) \cdot \hat{p}. \]  

(5)

This means that one neglects the term \( (e^2/2m)\hat{A}(r,t)^2 \) because it is quadratic correction with respect to the weak vector potential and one uses \( \hat{A}(0,0) \) instead of \( \hat{A}(r,t) \). The latter assumption, which corresponds to

\[ e^{i\mathbf{k} \cdot \mathbf{r}} = 1 + i\mathbf{k} \cdot \mathbf{r} + \frac{1}{2}(i\mathbf{k} \cdot \mathbf{r})^2 + \ldots \approx 1, \]  

(6)

is reliable if \( \mathbf{k} \cdot \mathbf{r} \ll 1 \), namely if the electromagnetic radiation has a wavelength \( \lambda = 2\pi/|\mathbf{k}| \) very large compared to the linear dimension \( R \) of the atom. Indeed, the approximation is fully justified in atomic physics where \( \lambda \approx 10^{-7} \) m and \( R \approx 10^{-10} \) m.
Quantum electrodynamics (I)

The total Hamiltonian of the matter-radiation system in the dipole approximation is then given by

\[ \hat{H} = \hat{H}_0 + \hat{H}_D , \]  

(7)

where

\[ \hat{H}_0 = \hat{H}_{\text{matt}} + \hat{H}_{\text{rad}} \]  

(8)

is the unperturbed Hamiltonian, such that

\[ \hat{H}_{\text{matt}} = \frac{\hat{p}^2}{2m} - \frac{e^2}{4\pi \epsilon_0 |r|} , \]  

(9)

is the matter Hamiltonian, while the radiation Hamiltonian reads

\[ \hat{H}_{\text{rad}} = \sum_k \sum_s \hbar \omega_k \hat{a}^+_k \hat{a}_k , \]  

(10)

where \( \hat{a}_k \) and \( \hat{a}^+_k \) are the annihilation and creation operators of the photon in the state \( |k_s\rangle \).
The eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ are of the form

$$|a\rangle |... n_{ks} ... \rangle = |a\rangle \otimes |... n_{ks} ... \rangle \quad (11)$$

where $|a\rangle$ is the eigenstate of $\hat{H}_{matt}$ with eigenvalue $E_a$ and $|... n_{ks} ... \rangle$ it the eigenstate of $\hat{H}_{rad}$ with eigenvalue $\sum_{k} \hbar \omega_k n_k$, i.e.

$$\hat{H}_0|a\rangle |... n_{kr} ... \rangle = \left( \hat{H}_{matt} + \hat{H}_{rad} \right) |a\rangle |... n_{ks} ... \rangle$$

$$= \left( E_a + \sum_{k} \hbar \omega_k n_k \right) |a\rangle |... n_{ks} ... \rangle \quad (12)$$