Lesson 2 - Quantum Properties of Light
Unit 2.2 Gas of photons and Planck law

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Structure of Matter - MSc in Physics
Let us consider the electromagnetic field in thermal equilibrium with a bath at the temperature $T$. The relevant quantity to calculate all thermodynamic properties of the system is the grand-canonical partition function $\mathcal{Z}$, given by

$$\mathcal{Z} = \text{Tr}[e^{-\beta(\hat{H} - \mu \hat{N})}]$$

where $\beta = 1/(k_B T)$ with $k_B = 1.38 \cdot 10^{-23}$ J/K the Boltzmann constant, $\hat{H} = \sum_k \sum_s \hbar \omega_k \hat{N}_{ks}$, is the quantum Hamiltonian without the zero-point energy,

$$\hat{N} = \sum_k \sum_s \hat{N}_{ks}$$

is the total number operator, and $\mu$ is the chemical potential, fixed by the conservation of the particle number.
Partition functions of photons (II)

For photons $\mu = 0$ and consequently the number of photons is not fixed. This implies that

$$Z = \sum_{\{n_{ks}\}} \langle \ldots n_{ks} \ldots | e^{-\beta \hat{H}} | \ldots n_{ks} \ldots \rangle$$

$$= \sum_{\{n_{ks}\}} \langle \ldots n_{ks} \ldots | e^{-\beta \sum_{k} \hbar \omega_{k} \hat{N}_{ks}} | \ldots n_{ks} \ldots \rangle$$

$$= \sum_{\{n_{ks}\}} e^{-\beta \sum_{k} \hbar \omega_{k} n_{ks}} = \sum_{\{n_{ks}\}} \prod_{ks} e^{-\beta \hbar \omega_{k} n_{ks}}$$

$$= \prod_{ks} \sum_{n_{ks}} e^{-\beta \hbar \omega_{k} n_{ks}} = \prod_{ks} \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_{k} n}$$

$$= \prod_{ks} \frac{1}{1 - e^{-\beta \hbar \omega_{k}}}.$$  \hspace{1cm} (4)
Quantum statistical mechanics dictates that the thermal average of any operator $\hat{A}$ is obtained as

$$\langle \hat{A} \rangle_T = \frac{1}{Z} \text{Tr}[\hat{A} e^{-\beta(\hat{H} - \mu \hat{N})}] . \quad (5)$$

In our case the calculations are simplified because $\mu = 0$. Let us suppose that $\hat{A} = \hat{H}$, it is then quite easy to show that

$$\langle \hat{H} \rangle_T = \frac{1}{Z} \text{Tr}[\hat{H} e^{-\beta \hat{H}}] = -\frac{\partial}{\partial \beta} \ln \left( \text{Tr}[e^{-\beta \hat{H}}] \right) = -\frac{\partial}{\partial \beta} \ln(Z) . \quad (6)$$

By using Eq. (4) we immediately obtain

$$\ln(Z) = -\sum_{k} \sum_{s} \ln \left( 1 - e^{-\beta \hbar \omega_k} \right) , \quad (7)$$

and finally from Eq. (6) we get

$$\langle \hat{H} \rangle_T = \sum_{k} \sum_{s} \frac{\hbar \omega_k}{e^{\beta \hbar \omega_k} - 1} = \sum_{k} \sum_{s} \hbar \omega_k \langle \hat{N}_{ks} \rangle_T . \quad (8)$$
Thermal energy of photons (II)

In the continuum limit, where

$$\sum_{\mathbf{k}} \rightarrow V \int \frac{d^3 \mathbf{k}}{(2\pi)^3},$$  \hspace{1cm} (9)

with $V$ the volume, and taking into account that $\omega_k = ck$, one can write the energy density $\mathcal{E} = \langle \hat{H} \rangle_T / V$ as

$$\mathcal{E} = 2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{c \hbar k}{e^{\beta \hbar c k} - 1} = \frac{\hbar c}{\pi^2} \int_0^\infty dk \frac{k^3}{e^{\beta \hbar c k} - 1},$$  \hspace{1cm} (10)

where the factor 2 is due to the two possible polarizations ($s = 1, 2$). By using $\omega = ck$ instead of $k$ as integration variable one gets

$$\mathcal{E} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta \hbar \omega} - 1} = \int_0^\infty d\omega \rho(\omega),$$  \hspace{1cm} (11)

where

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta \hbar \omega} - 1}$$  \hspace{1cm} (12)

is the energy density per frequency, i.e. the familiar formula of the black-body radiation, obtained for the first time in 1900 by Max Planck.
The previous integral can be explicitly calculated and it gives

\[ \mathcal{E} = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4, \quad \text{(13)} \]

which is nothing but the Stefan-Boltzmann law. In an similar way one determines the average number density of photons:

\[ n = \frac{\langle \hat{N} \rangle_T}{V} = \frac{1}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^2}{e^{\beta \hbar \omega} - 1} = \frac{2\zeta(3) k_B^3}{\pi^2 c^3 \hbar^3} T^3. \quad \text{(14)} \]

where \( \zeta(3) \approx 1.202 \). Notice that both energy density \( \mathcal{E} \) and number density \( n \) of photons go to zero as the temperature \( T \) goes to zero. We stress that these results are obtained at thermal equilibrium and under the condition of a vanishing chemical potential, meaning that the number of photons is not conserved when the temperature is varied.