Lesson 2 - Quantum Properties of Light
Unit 2.1 Fock states vs Coherent states

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Structure of Matter - MSc in Physics
Quantum electromagnetic field (1)

We have seen that the quantum Hamiltonian of the light can be then written as

\[ \hat{H} = \sum_k \sum_s \hbar \omega_k \left( \hat{N}_{ks} + \frac{1}{2} \right). \]  

(1)

Here

\[ \hat{N}_{ks} = \hat{a}_{ks}^+ \hat{a}_{ks} \]  

(2)

is the single-mode number operator, which counts the number of photons in the single-mode state \( |ks\rangle \), with \( k \) the wavevector and \( s = 1, 2 \) the polarization.

The quantum electric and magnetic fields can be obtained from the classical expressions. In this way we obtain

\[ \hat{E}(r, t) = i \sum_k \sum_s \sqrt{\frac{\hbar \omega_k}{2 \varepsilon_0 V}} \left[ \hat{a}_{ks} e^{i(k \cdot r - \omega_k t)} - \hat{a}_{ks}^+ e^{-(i k \cdot r - \omega_k t)} \right] \varepsilon_{ks}, \]  

(3)

\[ \hat{B}(r, t) = \sum_k \sum_s \sqrt{\frac{\hbar}{2 \varepsilon_0 \omega_k V}} \left[ \hat{a}_{ks} e^{i(k \cdot r - \omega_k t)} - \hat{a}_{ks}^+ e^{-i(k \cdot r - \omega_k t)} \right] i \mathbf{k} \wedge \varepsilon_{ks}. \]  

(4)
Let us now consider for simplicity a linearly polarized monochromatic wave of the radiation field with wavevector \( \mathbf{k} \) and polarization \( s \). One finds immediately that the quantum electric field can be then written in a simplified notation as

\[
\hat{E}(r, t) = \sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} i \left( \hat{a} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - \hat{a}^+ e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right) \varepsilon
\]

(5)

where \( \omega = \omega_k = c|\mathbf{k}| \). Notice that, to simplify the notation, we have removed the subscripts in the annihilation and creation operators \( \hat{a} \) and \( \hat{a}^+ \).

If there are exactly \( n \) photons in this polarized monochromatic wave the Fock state of the system is given by

\[
|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle.
\]

(6)
Quantum electromagnetic field (III)

It is then straightforward to show that

$$\langle n | \hat{E}(r, t) | n \rangle = 0,$$

for all values of the photon number $n$, no matter how large. This result holds for all modes, which means then that the expectation value of the electric field in any many-photon Fock state is zero. On the other hand, the expectation value of $\hat{E}(r, t)^2$ is given by

$$\langle n | \hat{E}(r, t)^2 | n \rangle = \frac{\hbar \omega}{\varepsilon_0 V} \left( n + \frac{1}{2} \right).$$

Obviously a similar reasoning applies for the magnetic field (4).
The strange result of Eq. (7) is due to the fact that the expectation value is performed with the Fock state $|n\rangle$, which means that the number of photons is fixed because

$$\hat{N}|n\rangle = n|n\rangle . \quad (9)$$

Nevertheless, usually the number of photons in the radiation field is not fixed, in other words the system is not in a pure Fock state. For example, the radiation field of a well-stabilized laser device operating in a single mode is described by a coherent state $|\alpha\rangle$, such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle , \quad (10)$$

with

$$\langle \alpha|\alpha \rangle = 1 . \quad (11)$$
The coherent state $|\alpha\rangle$, introduced in 1963 by Roy Glauber, is thus the eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue $\alpha = |\alpha| e^{i\theta}$. $|\alpha\rangle$ does not have a fixed number of photons, i.e. it is not an eigenstate of the number operator $\hat{N}$, and it is not difficult to show that $|\alpha\rangle$ can be expanded in terms of number (Fock) states $|n\rangle$ as follows

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (12)$$

From Eq. (10) one immediately finds

$$\bar{N} = \langle \alpha | \hat{N} | \alpha \rangle = |\alpha|^2, \quad (13)$$

and it is natural to set

$$\alpha = \sqrt{\bar{N}} e^{i\theta}, \quad (14)$$

where $\bar{N}$ is the average number of photons in the coherent state, while $\theta$ is the phase of the coherent state.
We observe that the coherent state $|\alpha\rangle$ is such that

$$\langle \alpha | \hat{N}^2 | \alpha \rangle = |\alpha|^2 + |\alpha|^4 = \bar{N} + \bar{N}^2$$

(15)

and consequently

$$\langle \alpha | \hat{N}^2 | \alpha \rangle - (\langle \alpha | \hat{N} | \alpha \rangle)^2 = \bar{N},$$

(16)

while

$$\langle n | \hat{N}^2 | n \rangle = n^2$$

(17)

and consequently

$$\langle n | \hat{N}^2 | n \rangle - (\langle n | \hat{N} | n \rangle)^2 = 0.$$
Coherent states (IV)

It is now easy to prove that the expectation value of the electric field \( \hat{E}(r, t) \) of the linearly polarized monochromatic wave, Eq. (5), in the coherent state \( |\alpha\rangle \) reads

\[
\langle \alpha | \hat{E}(r, t) | \alpha \rangle = -\sqrt{\frac{2\tilde{N}\hbar \omega}{\varepsilon_0 V}} \sin(k \cdot r - \omega t + \theta) \varepsilon ,
\]

(19)

while the expectation value of \( \hat{E}(r, t)^2 \) is given by

\[
\langle \alpha | \hat{E}(r, t)^2 | \alpha \rangle = \frac{2\tilde{N}\hbar \omega}{\varepsilon_0 V} \sin^2(k \cdot r - \omega t + \theta) + \frac{\hbar \omega}{2\varepsilon_0 V} .
\]

(20)

These results suggest that the coherent state is indeed a useful tool to investigate the correspondence between quantum field theory and classical field theory.