Lesson 34 - 19/12/2022

A "jump" into Hamiltonian formalism

- Basic idea of Hamiltonian formalism: use (q,p) instead of $(q,\dot q)$. With this idea, if $L(q,\dot q)$ has q_e as cyclic coordinate then $p_e := \frac{\partial L}{\partial \dot q_e}$ is a conserved coordinate. $(\dot p_e = 0)$.
- Moreover, Hamilton eqs are of first order (see below). (Recall that E-L eqs are of 2nd order).

→ **Legendre transformation**

$\mathcal{U} \subseteq \mathbb{R}^n$ open set

Let $L : \mathcal{U}\times\mathbb{R}^n \longrightarrow \mathbb{R}$
$(q,\dot q) \mapsto L(q,\dot q)$

$$\Lambda_L = \text{Legendre transform} : \begin{array}{ccc} \mathcal{U}\times\mathbb{R}^n & \longrightarrow & \mathcal{U}\times\mathbb{R}^n \\ (q,\dot q) & \longmapsto & (q, p = \frac{\partial L}{\partial \dot q}(q,\dot q)) \end{array}$$

$= (q_1, q_2, \dots, q_n)$

Let fix a point $(\bar q, \bar v) \in \mathcal{U}\times\mathbb{R}^n$ and use the inverse function theorem.

if $\frac{\partial \Lambda_L}{\partial (q,\dot q)}(\bar q,\bar v)$ is invertible then

$(q,\dot q) \mapsto (q, p = \frac{\partial L}{\partial \dot q}(q,\dot q))$ is a <u>local</u> diffeo. around $(\bar q, \bar v)$.

$$\frac{\partial \Lambda_L}{\partial (q,\dot q)}(\bar q,\bar v) = \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \frac{\partial^2 L}{\partial q\,\partial \dot q}(\bar q,\bar v) & \frac{\partial^2 L}{\partial \dot q\,\partial \dot q}(\bar q,\bar v) \end{pmatrix}$$

is invertible if $\det\left(\frac{\partial^2 L}{\partial \dot q\,\partial \dot q}(\bar q,\bar v)\right) \neq 0$.

↙

Same condition assuring that Lagrange eqs can be written in normal form!

In the mechanical case (+ eventually a "generalized" potential)

$$L(q,\dot q) = \frac{1}{2}\langle \dot q, a(q)\,\dot q\rangle - V(q) - \underbrace{V_1(q,\dot q)}_{= -b(q)\dot q}$$

$$= \frac{1}{2}\langle \dot q, a(q)\dot q\rangle - V(q) + b(q)\dot q$$

$$\Lambda_L : (q,\dot q) \longmapsto (q, \underbrace{p = a(q)\dot q + b(q)}_{\downarrow})$$

$$p - b(q) = a(q)\dot q$$

with ~~global~~ inverse

$$(q,p) \longmapsto (q, \underbrace{a^{-1}(q)(p-b(q))}_{\substack{\shortparallel \\ v(q,p)}})$$

Proposition

Λ_L conjugates (= sends solutions into solutions) the E-L eqs of Lagrangian $L(q,\dot q)$ into Hamilton eqs

$$\begin{cases} \dot q_i = \dfrac{\partial H}{\partial p_i}(q,p) \\[2ex] \dot p_i = -\dfrac{\partial H}{\partial q_i}(q,p) \end{cases} \qquad \forall\ i = 1 \_ n$$

(*)

of Hamiltonian $H(q,p) = \left[\sum_{J=1}^{n} p_J \dot q_J - L(q,\dot q) \right]\Big|_{\dot q = v(q,p)}$

$$\left[\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix} \right.$$

$x = \begin{pmatrix} q \\ p \end{pmatrix} \iff \dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(x)$]

Proof: we need to prove that

$t \mapsto q(t)$ solves E-L eqs for L $\iff$

$t \mapsto (q(t), p(t)) = \Lambda_L(q(t), \dot{q}(t))$

solves the Hamilton eqs for H.

$$\frac{\partial H}{\partial q_i} = \cancel{\sum_{J=1}^{n} p_J \frac{\partial v_J}{\partial q_i}} - \frac{\partial L}{\partial q_i} - \cancel{\sum_{J=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_J}}_{= p_J} \frac{\partial v_J}{\partial q_i}} = -\frac{\partial L}{\partial q_i}$$

$$\frac{\partial H}{\partial p_i} = v_i + \cancel{\sum_{J=1}^{n} p_J \frac{\partial v_J}{\partial p_i}} - \cancel{\sum_{J=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_J}}_{= p_J} \frac{\partial v_J}{\partial p_i}} = v_i$$

($\Rightarrow$)

Let $t \mapsto q(t)$ solving E-L eqs for L.

Now, by E-L eqs and def. of conj. momenta:

$$\dot{q}(t) = v(q(t), p(t)) = \frac{\partial H}{\partial p}(q(t), p(t))$$

$$\dot p(t) = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot q}(q(t), \dot q(t))\right] = -\frac{\partial H}{\partial q}(q(t), \dot q(t))$$

$$\underset{\text{E-L eqs}}{\underset{\uparrow}{=}} \frac{\partial \mathcal{L}}{\partial q}(q(t), \dot q(t)) \stackrel{\sim}{=}$$

that is

$$\begin{cases} \dot q(t) = \dfrac{\partial H}{\partial p}(q(t), p(t)) \\[2ex] \dot p(t) = -\dfrac{\partial H}{\partial q}(q(t), p(t)) \end{cases}$$

which means that

$t \mapsto (q(t), p(t))$ solves Hamilton eqs for H (def. as $(\star)$.)

$(\Leftarrow)$ Same argument. $\boxtimes$

- WEDN. 21/12/2022

Talk by F. Cardin on

Levi-Civita and the parallel transport

- LAST LECTURE
  MONDAY 09/01/2023
  At 10:30 EF 5

- FIRST TOTAL EXAM + 2ND PARTIAL EXAM
  FRIDAY 20/01/2023
  At 10:30 ROOM ?

THE MECHANICAL CASE

$$\Lambda_L : (q, \dot q) \longmapsto (q, \overbrace{a(q)\dot q}^{p = })$$

$$\Lambda_L^{-1} : (q, p) \longmapsto (q, \underbrace{a^{-1}(q)\, p}_{= v(q,p)})$$

$$H(q,p) = \langle p, a^{-1}(q)p \rangle - \frac{1}{2}\langle a^{-1}(q)\,p, \underbrace{a(q)\, a^{-1}(q)\, p}_{= \mathbb{1}} \rangle + V(q)$$

$$= \frac{1}{2} \langle p, a^{-1}(q)\, p \rangle + V(q)$$

Suppose that $a(q) = \mathbb{I}$

$$L = \frac{1}{2} |\dot{q}|^2 - V(q)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

EQUIVALENT by Λ_L

$$\longleftrightarrow$$

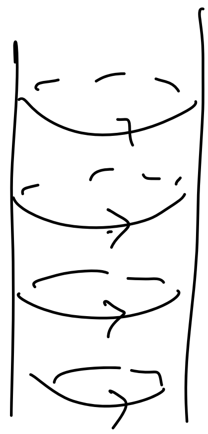
$$H = \frac{1}{2} |p|^2 + V(q)$$

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\nabla V(q) \end{cases}$$

In the Hamiltonian formalism plays an important role the integrable Hamiltonian

$$K(p) = \frac{|p|^2}{2} + \varepsilon V(q)$$

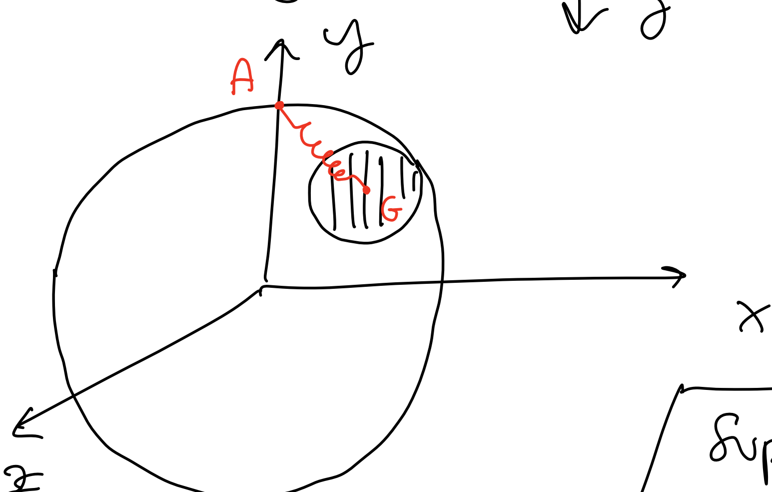
$$\begin{cases} \dot q = p \\ \dot p = 0 \end{cases}$$



— x — x —

Oxyz uniformly rotating with

$\vec{\omega} = \omega \hat{y}$ $\qquad \downarrow \vec{g}$

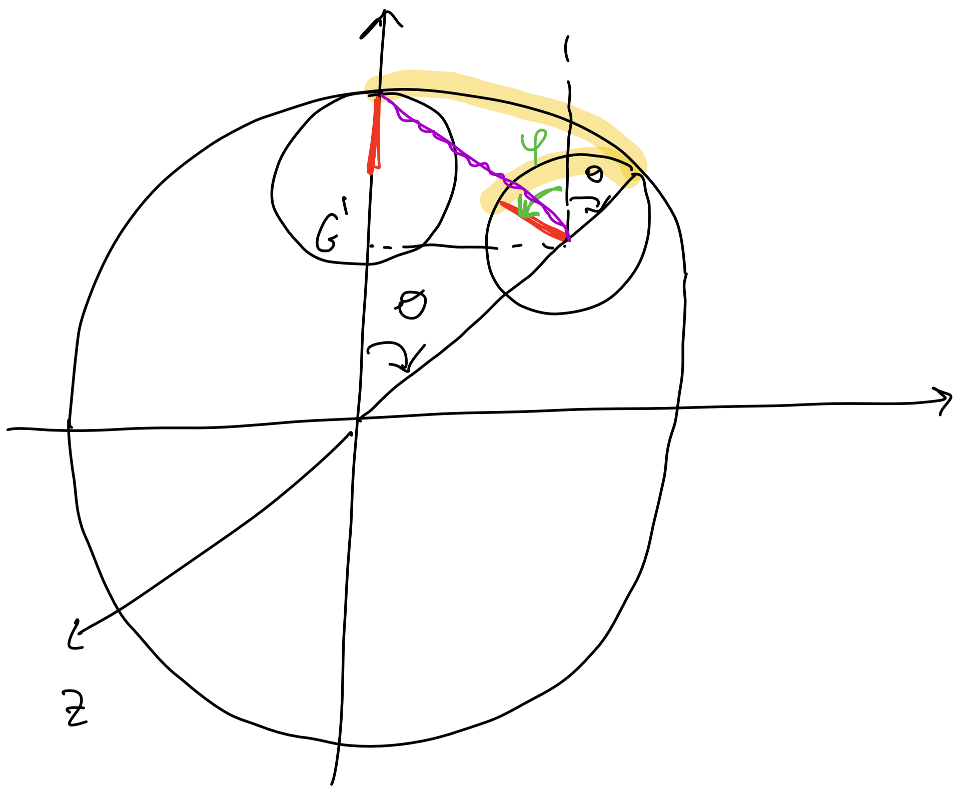


$R > r$

$h > 0$

DISC m, r

$\boxed{\text{Suppose } hR - mg > 0}$



θ = Lagrangian parameter

$$\vec{\omega}_D = \dot{\varphi}\,\hat{z}$$

$$r(\varphi + \theta) = R\theta$$

$$\varphi = \frac{(R - r)}{r}\,\theta$$

$$\dot{\varphi} = \frac{R - r}{r}\,\dot{\theta}$$

$$\vec{\omega}_D = \frac{R-r}{r}\,\dot{\theta}\,\hat{z}$$

$$K(\theta, \dot{\theta}) =$$

$$= \frac{1}{2} m |\vec{v}_G|^2 + \frac{1}{2}\left(\frac{m r^2}{2}\right)\left(\frac{R-r}{r}\right)^2 \dot{\theta}^2$$

$$|\vec{v}_G| = r\dot{\varphi} =$$

$$= \cancel{r}\,\frac{R-r}{\cancel{r}}\,\dot{\theta} = (R-r)\,\dot{\theta}$$

$$\boxed{K(\theta, \dot{\theta}) =}$$

$$= \frac{1}{2} m (R-r)^2 \dot{\theta}^2 +$$

$$+ \frac{1}{2}\,\frac{m\cancel{r^2}}{2}\,\frac{(R-r)^2}{\cancel{r^2}}\,\dot{\theta}^2$$

$$= \frac{3}{4} m (R-r)^2 \dot{\theta}^2$$

Potential energy.
(Recall that the work referred to the Coriolis force $\equiv 0$)

$U(\theta) =$

$= mg(R-r)\cos\theta +$
$+ \frac{h}{2}\left[|AG'|^2 + |G'G|^2\right] -$
$- \frac{\omega^2 m}{2}\left[|GG'|^2\right] + \text{consts}.$

$U(\theta) = mg(R-r)\cos\theta +$
$+ \frac{h}{2}\left\{\left[R-(R-r)\cos\theta\right]^2 + (R-r)^2\sin^2\theta\right\} -$

$- \frac{\omega^2 m}{2}(R-r)^2\sin^2\theta$

$U(\theta) = mg(R-r)\cos\theta - hR(R-r)\cos\theta$
$- \frac{\omega^2 m}{2}(R-r)^2\sin^2\theta + \text{consts}.$

Equilibria

$u'(\theta) =$

$$-mg\,\cancel{(R-r)}\sin\theta + hR\,\cancel{(R-r)}\sin\theta$$

$$-\omega^2 m (R-r)^{\cancel{2}} \sin\theta\cos\theta = 0$$

$$\sin\theta\left[-mg + hR - \omega^2 m\,(R-r)\cos\theta\right] = 0$$

$\theta_1 = 0$

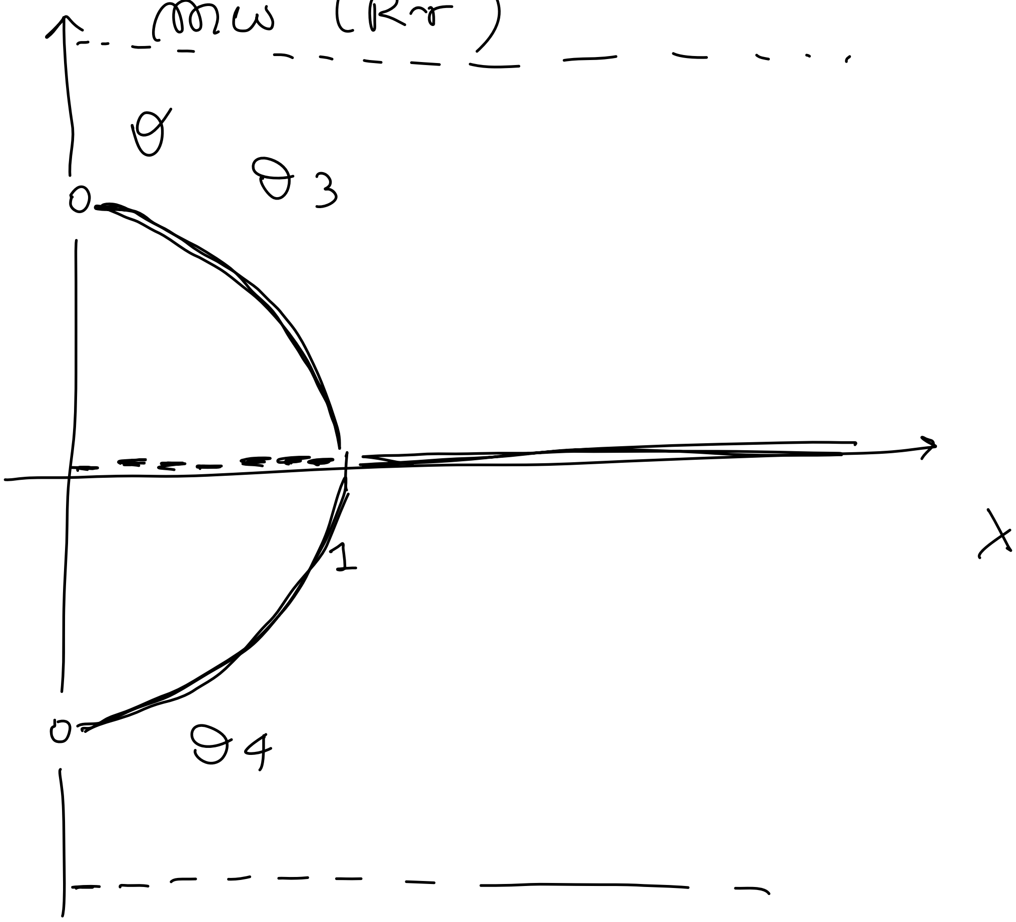
$\theta_2 = \pi$

AND if $\dfrac{hR - mg}{m\omega^2(R-r)} < 1$

$\exists\ \theta_3 : \arccos\left(\dfrac{hR - mg}{m\omega^2(R-r)}\right)$

and $\theta_4 = -\theta_3$

$$\lambda = \frac{hR - mg}{m\omega^2 (R-r)}$$



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