Lesson 33 - 15/12/2022

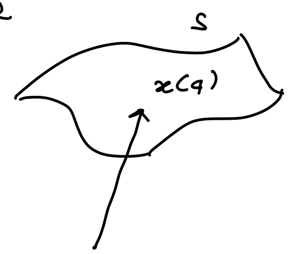
## Spontaneous motions & geodesics on a surface.

$\downarrow$

$L = K$

Let $S \subseteq \mathbb{R}^3$, $\dim = 2$, the constraint of a particle of mass $m > 0$.

$$S \hookrightarrow \mathbb{R}^3, \quad q = (q_1, q_2) \mapsto x(q) \in \mathbb{R}^3$$




- Suppose that there are not external forces. Then spontaneous motions between $q_0 \neq q_1$ in the interval $[t_0, t_1]$ $(t_0 < t_1)$ are (by Hamilton's principle) exactly the critical curves of the corresponding action functional:

$$J_{L=K}[q(\cdot)] = \int_{t_0}^{t_1} K(q(t), \dot{q}(t))\, dt$$

Spontaneous motions

$\longrightarrow K \nearrow$

where $$K(q, \dot{q}) = \frac{m}{2} \sum_{i=1}^{3} \sum_{h,k=1}^{2} \frac{\partial x_i}{\partial q_h}(q) \frac{\partial x_i}{\partial q_k}(q)\, \dot{q}_h \dot{q}_k$$

- Now we take into account another (purely geometrical) problem. For the same $S \hookrightarrow \mathbb{R}^3$, the same $q_0 \neq q_1$ and the same interval $[t_0, t_1)$ (now $t$ is not necessary the time!), let consider the lenght functional:

$$\ell : \Gamma_{t_0,t_1}^{q_0,q_1} \longrightarrow \mathbb{R}$$

$$\ell[q(\cdot)] = \int_{t_0}^{t_1} \left| \frac{dx}{dq}(q(t)) \right|_{\mathbb{R}^3} dt =$$

$$= \int_{t_0}^{t_1} \left( \left[ \sum_{i=1}^{3} \sum_{h=1}^{2} \frac{\partial x_i}{\partial q_h}(q(t))\, \dot{q}_h \right] \left[ \sum_{i=1}^{3} \sum_{k=1}^{2} \frac{\partial x_i}{\partial q_k}(q(t))\, \dot{q}_k \right] \right)^{1/2} dt$$

$$= \int_{t_0}^{t_1} \sqrt{\frac{2}{m} K(q(\cdot), \dot{q}(t))}\, dt$$

$\searrow \sqrt{\frac{2}{m} K} \longrightarrow$ geodesics

Natural question: which is the relation between the two problems?

**Prop** If $q(\cdot) \in \Gamma^{q_0, q_1}_{t_0, t_1}$ solves E-L equations for $L = k$, then $q(\cdot)$ solves E-L equations for $L = \sqrt{\frac{2}{m} k}$.

In other words, spontaneous motions ($L = k$) on $S$ are geodesics on $S$.

**Proof** Along the motion given by $q(\cdot)$, $L = k$ is a conserved quantity. Moreover, $q_0 \neq q_1$, $k > 0$.

$$\frac{d}{dt} \frac{\partial \sqrt{\frac{2}{m} k}}{\partial \dot{q}_i} - \frac{\partial \sqrt{\frac{2}{m} k}}{\partial q_i} =$$

$$= \frac{d}{dt} \left( \frac{1}{\cancel{2}\sqrt{\frac{2k}{m}}} \cdot \frac{\cancel{2}}{m} \frac{\partial k}{\partial \dot{q}_i} \right) - \frac{1}{\sqrt{\frac{2k}{m}}} \cdot \frac{1}{m} \frac{\partial k}{\partial q_i} =$$

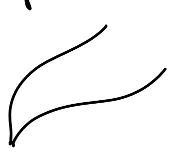
$$= \frac{1}{\sqrt{2mk}} \underbrace{\left[ \frac{d}{dt} \frac{\partial k}{\partial \dot{q}_i} - \frac{\partial k}{\partial q_i} \right]}_{} = 0 \qquad \square$$

Other direction? It's true. In particular, given a geodesic (curve solving the E-L eqs. for $L = \sqrt{\frac{2k}{m}}$), can I find a corresponding spontaneous motion?

Some remarks

1) If $t \mapsto q(t)$ is a geodesic with $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$, we can find infinitely many parameterizations $t = t(\tau)$ starting from the same initial conditions.
This "degeneration" comes from the fact that E-L eqs. for $\sqrt{\frac{2k}{m}}$ cannot be written in normal form.

2) However, fixed $q_0, \dot{q}_0$, it is not possible to find solutions for E-L for $\sqrt{\frac{2k}{m}}$ with different images.



③ Finally, given a geodesic $t \mapsto q(t)$, $\exists$ a parametrization $\tau \mapsto q(t(\tau))$ solving E-L equations for K.

"Geodesics on a mnf $S \subseteq \mathbb{R}^3$ correspond to free motions"

- GEODESICS on the PLANE

Consider a curve $\gamma(t)$ on $\mathbb{R}^2$ given by $t \mapsto \begin{cases} t \\ u(t) \end{cases}$

$a \leq t \leq b$.

then $\ell[\gamma(t)] = \int_a^b \underbrace{\sqrt{1 + [u'(t)]^2}} \, dt$

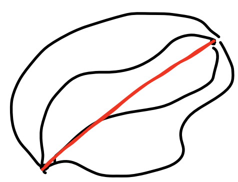
$L(u, u') \longrightarrow$ has a cyclic coo.: $u$

$\Rightarrow \exists$ conserved quantity $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u'} = 0$

That is $\frac{\partial \mathcal{L}}{\partial u'}$ a first integral.

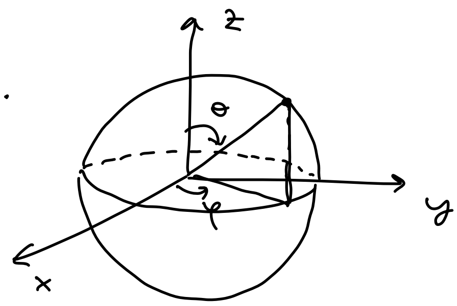
$\frac{\partial \mathcal{L}}{\partial u'} = \frac{1}{\cancel{2}\sqrt{1+(u')^2}} \cdot \cancel{2} u' \Rightarrow u'$ is constant.

Geodesics are straight lines.



- GEODESICS on the SPHERE

We use spherical coordinates.




$$\begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases}$$

$L(\theta, \varphi, \dot\theta, \dot\varphi) = R^2(\dot\theta^2 + \dot\varphi^2 \sin^2\theta)$

We divide by $2R^2$ and consider the Lagrangian

$$\boxed{\frac{1}{2}\dot\theta^2 + \frac{1}{2}\sin^2\theta \, \dot\varphi^2}$$

$\sin^2\theta \, \dot\varphi = J \qquad (J \neq 0)$

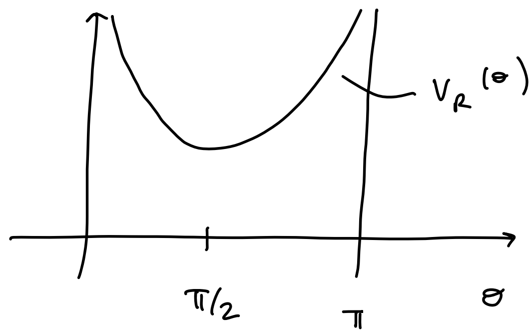
$\dot\varphi = J / \sin^2\theta$

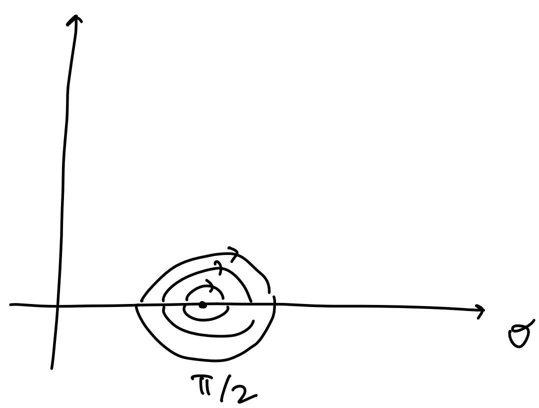
$\varphi$ is a cyclic coo. $\Rightarrow$ we can use the reduced Lagrangian

$$L_R(\theta, \dot\theta) = \frac{1}{2}\dot\theta^2 + \frac{1}{2}\cancel{\sin^2\theta}\,\frac{J^2}{\sin^4\theta} - J\,\frac{J}{\sin^2\theta}$$

$$= \frac{1}{2}\dot\theta^2 - \frac{1}{2}\frac{J^2}{\sin^2\theta}$$

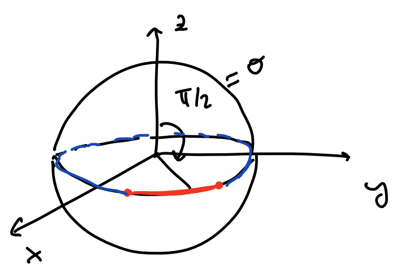
Reduced potential energy: $V_R(\theta) = \dfrac{1}{2}\dfrac{J^2}{\sin^2\theta}$



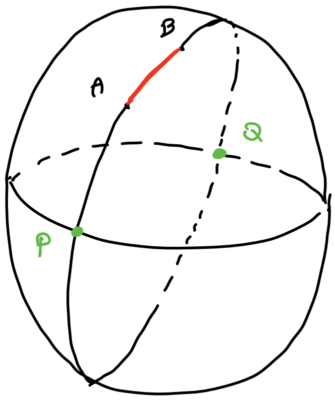

Re-construction of the dynamics for $\theta = \pi/2$.

$\theta = \pi/2$, $\varphi = \varphi_0 + Jt$ $\quad (J = \dot\varphi_0)$




↓ ARCS of equator are GEODESICS.

More generally: GEODESICS on the SPHERE ARE CYCLES ON THE SPHERE WHOSE CENTERS COINCIDE WITH THE CENTER OF THE SPHERE (Great CIRCLES)

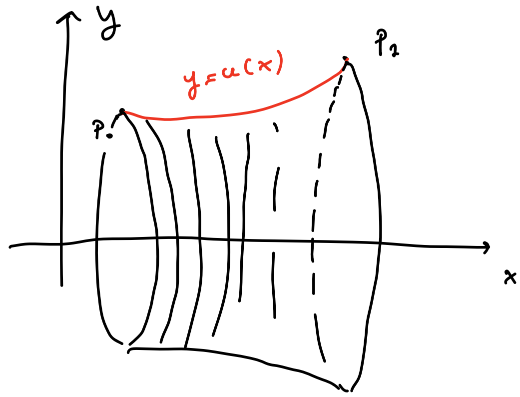



- Through any 2 points on a sphere, not opposite each other (not antipodal) (as A, B) $\exists !$ great circle.

- Through 2 antipodal points $\exists$ infinitely many great cycles (as P, Q).

EXERCISE (on Calculus of Variations)

$Oxy$, $y = u(x)$ $(u \in C^{\infty}(\mathbb{R},\mathbb{R}))$ through 2 fixed points $P_0, P_1$.




$S[u]$ = area of the revolution surface generated by $u$.

Which is the function $u$ which minimizes $S[u]$ ?!

$\boxed{1}$ Arc from $x$ to $x+dx$ has lenght $\sqrt{1+(u'(x))^2}\,dx$

$$S[u] = 2\pi \int_{x_0}^{x_1} \underbrace{u(x)\sqrt{1+[u'(x)]^2}}\,dx$$

$$L(u,u') = u\sqrt{1+(u')^2}$$

$$\text{E-L}: \quad \frac{d}{dx}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u} = 0$$

$$\frac{\partial L}{\partial u'} = \frac{u\cdot \cancel{2}u'}{\cancel{2}\sqrt{1+(u')^2}} = \frac{uu'}{\sqrt{1+(u')^2}}$$

$$\frac{d}{dx}\frac{\partial L}{\partial u'} = \frac{\sqrt{1+(u')^2}\left((u')^2+uu''\right) - \frac{1}{\cancel{2}\sqrt{1+(u')^2}}\cdot \cancel{2}u'u''\,uu'}{1+(u')^2}$$

$$= \frac{(1+(u')^2)\left((u')^2+uu''\right) - (u')^2 uu''}{(1+(u')^2)^{3/2}} =$$

$$= \frac{(u')^2 + uu'' + (u')^4 + \cancel{uu''(u')^2} - \cancel{(u')^2 uu''}}{(1+(u')^2)^{3/2}}$$

E-L equations.

$$\frac{\partial \mathcal{L}}{\partial u} = \sqrt{1+(u')^2}$$

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'} - \frac{\partial \mathcal{L}}{\partial u} =$$

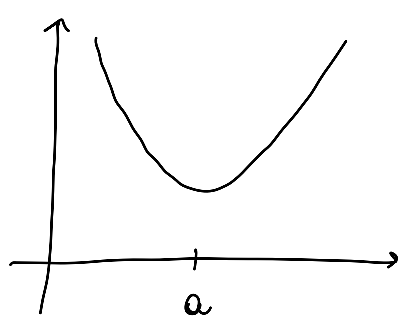
$$= \frac{(u')^2 + uu'' + (u')^4}{(1+(u')^2)^{3/2}} - \sqrt{1+(u')^2} = 0$$

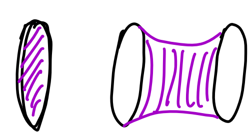
$$\Leftrightarrow (u')^2 + uu'' + (u')^4 - (1+(u')^2)^2 = 0$$

$$\Leftrightarrow (u')^2 + uu'' + \cancel{(u')^4} - 1 - \cancel{(u')^4} - 2(u')^2 = 0$$

$$\Leftrightarrow \boxed{uu'' = 1 + (u')^2}$$

Solved by $u(x) = h \cosh\left(\frac{x-a}{h}\right) \rightarrow$ catenary



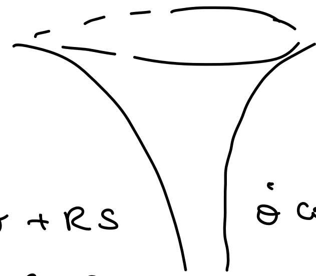



$\boxed{\text{EX}}$

$$(s,\theta) \longmapsto \left(Rs\cos\theta,\ Rs\,\mathrm{sen}\,\theta,\ -R/s\right)$$

$s>0$

$\downarrow \vec{g}$



$$\vec{OP} = \left(Rs\cos\theta,\ Rs\sin\theta,\ -R/s\right)$$

$$\vec{v}_p = \left(R\dot{s}\cos\theta - Rs\dot{\theta}\sin\theta,\ R\dot{s}\sin\theta + Rs\dot{\theta}\cos\theta,\ \frac{R\dot{s}}{s^2}\right)$$

$$|\vec{v}_p|^2 = R^2\dot{s}^2 + R^2 s^2\dot{\theta}^2 + \frac{R^2\dot{s}^2}{s^4}$$

$$K = \frac{1}{2} m R^2 \left(\dot{\theta}^2 s^2 + \dot{s}^2\left(\frac{1+s^4}{s^4}\right)\right)$$

$$V = -\frac{mgR}{s} \qquad L = K - V$$

$\theta$ cyclic coo.

$$J = \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta}s^2 \Rightarrow \qquad \dot{\theta} = \frac{J}{mR^2s^2}$$

$$L_R(s,\dot{s}) = \dots = \frac{1}{2} mR^2\left(\frac{1+s^4}{s^4}\right)\dot{s}^2 - \underbrace{\left(\frac{J^2}{2mR^2s^2} - \frac{gmR}{s}\right)}_{V_R(s)}$$

$$V'_R(s) = 0 \iff$$

$$-\frac{\cancel{2}J^2\cancel{s}}{\cancel{2}mR^2s^{\cancel{4}\,3}} + \frac{gmR}{s^2} = 0$$

$$\iff \frac{1}{s^2}\left(\frac{J^2}{mR^2s} - gmR\right) = 0 \iff \bar{s} = \frac{J^2}{m^2gR^3}$$

$V''_R(\bar{s}) = \; > 0 \implies$ minimum : stable!

For the original system:

$$S_t \equiv J^2 / m^2 g R^3 = \bar{s}$$

$$\theta_t = \theta_0 + \frac{g^2 m^3 R^4}{J^3} t$$

—— x —— x ——