Lie groups and Symmetry - Course Evaluation
Doctoral School in Mathematics

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Exam topics

The examination will consist of an interview in which you will be asked to present and answer questions on three different topics chosen at random from each of the following three groups of topics.

Group 1

Topic 1 Definition and examples of Lie groups. Structure as a Lie group of $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $SL(n,\mathbb{R})$ and their main topological properties (dimension, connected, non-connected, compact, non-compact, etc if discussed in class). Proof that $O(n)$ and $SO(n)$ are compact and $U(n)$ is connected. Left and right translations and conjugation (*Reference:* [1, Sections 1.1.A, 1.1.B, 1.1.C].)

Topic 2 Left and right translations and conjugation. Definition of Lie group homomorphisms and Lie group isomorphisms. Proof that Lie group homomorphisms have constant rank. Definition and examples of Lie subgroups. Difference between closed and non-closed Lie subgroups and examples of both. (*Reference:* [1, Sections 1.1.C, 1.1.D, 1.1.E].)

Topic 3 Definition and examples of Lie algebras and subalgebras. Definition of Lie algebra homomorphisms and isomorphisms. $(\mathbb{R}^3,\times)$ is isomorphic to $\mathrm{skew}(3)$. Left invariant vector fields and their properties. Definition of the Lie algebra of a Lie group using left invariant vector fields. Proof that the Lie algebra of an abelian Lie group is abelian. (*Reference:* [1, Sections 1.2.A, 1.2.B, 1.2.C, 1.2.D].)

Group 2

Topic 1 Definition of the exponential map and its main properties. Proof that if $\Psi : G \to H$ is a Lie group homomorphism (isomorphism) then $T_{e_g}\Psi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism (isomorphism) and the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{g} & \xrightarrow[T_{e_G}\Psi]{} & \mathfrak{h} \end{array}$$

Examples: the exponential map in $(\mathbb{R}^n,+)$, S^1, $\mathbb{T}^n$ and $GL(n,\mathbb{R})$. (*Reference:* [1, Sections 1.2.E, 1.2.F, 1.3.A, 1.3.B].)

Topic 2 Definition and main properties of the Lie group and Lie algebra adjoint representations Ad and ad. Proof that $\mathrm{ad}_\xi \eta = [\![\xi, \eta]\!]$. Calculation of the Lie bracket in $L(n,\mathbb{R})$, the Lie algebra of $GL(n,\mathbb{R})$. (*Reference:* [2, Section 1].)

Topic 3 The Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G can be interpreted as a subalgebra of the Lie algebra $\mathfrak{g}$ of G and with this interpretation $\exp_H = \exp_G|_{\mathfrak{h}}$. Application to determine the Lie algebra and exponential maps of $O(n)$, $SO(n)$, $SL(n,\mathbb{R})$ and $U(n)$ viewing them as Lie subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. One parameter subgroups. (*Reference:* [1, Sections 1.2.F, 1.3.A, 1.3.B, 1.3.C].)

Group 3

Topic 1 Lie group actions. Definitions, main properties and examples. Definition of free, transitive and proper actions. Proof that every $\mathbb{R}$-action is the flow of a vector field. Definition and main properties of the isotropy subgroup. The quotient space and its induced topology. Properties of the orbit space for free and proper actions. Examples of non-free/non-proper actions whose orbit space does not possess a differentiable structure. Definition, main properties and examples of the infinitesimal generators of a Lie group action. Proof that for a free action all non-trivial infinitesimal generators are everywhere non-vanishing. Examples. (*Reference:* [1, Sections 2.1.A, 2.1.B, 2.1.D, 2.2.B].)

Topic 2 Definition, main properties and examples of the infinitesimal generators of a Lie group action. Examples. Proof that the group orbits of a free action are immersed submanifolds diffeomorphic to G. Invariant vector fields. Proof that if H is a Lie subgroup of G then $\mathrm{Fix}(H)$ is invariant under the flow of any G-invariant vector field. Application to $SO(2)$-invariant vector fields in $\mathbb{R}^2$. Reduction of G-invariant vector fields under the assumption that M/G is a smooth manifold and the orbit map $\pi : G \to M/G$ is a submersion. Examples. (*Reference:* [1, Sections 2.2.A, 2.2.B].)

Topic 3 Definition of relative equilibrium (RE) of a G-invariant vector field X. Angular velocity ξ of a RE. Proof that for a RE m_0 with angular velocity ξ it holds $\Phi_t^X(m_0) = \Psi_{\exp(\xi t)}(m_0)$. Examples. Dynamics along the RE of free actions of compact Lie groups. (*Reference:* [2, Section 3].)

Background in differential geometry

A good acquaintance with the following topics from differential geometry is essential to present the examination.

- Submanifolds and immersed submanifolds.
- Immersions and submersions.
- If $f : M \to N$ is a submersion then $f^{-1}(\{n\})$ is a smooth manifold and for $m \in f^{-1}(\{n\})$ one has $T_m f^{-1}(\{n\}) = \ker T_m f$.
- Diffeomorphisms and local diffeomorphisms.
- The chain rule.

- Vector fields
  - Flow and its main properties.
  - Lie derivative of functions along a vector fields. Characterisation of vector fields as derivations.
  - Lie bracket of vector fields. Interpretation of commutativity of vector fields in terms of their flows.
  - Pull-back and push-forward of vector fields by diffeomorphisms and their main properties.
  - Lie derivative of a vector field Y along a vector field X and the relationship with the Lie bracket.
  - φ-related vector fields by a smooth map φ between two manifolds.

This material was reviewed during the lectures and is also reviewed at different points in [1] and its Appendix.

Alternative evaluation

As an alternative to the evaluation scheme presented above (which is the preferred one) you may choose to focus your attention on the relation between Lie algebras and Lie groups (*References* [2, Section 2], [1, Section 1.3.D]), and give a presentation of this material complementing it with a proof of [2, Theorem 2.2 or Theorem 2.4]. If you choose to proceed in this way, you should be able to prove statements or answer questions from:

(i) Any material of differential geometry not covered in the list "Background in differential geometry" that you use in the proofs of [2, Theorem 2.2 or Theorem 2.4].

(ii) Topic 3 of Group 1 above.

(iii) Topics 1 and 2 of Group 2 above.

If due to your background/research interests, you have serious problems to take the evaluation with both of the two proposed schemes, please contact me.

References

[1] Fassò F. *Notes on Lie groups and symmetry 2022*.

[2] García-Naranjo L.C. *Notes on "Extra material presented on class"*.