Lesson 32 - 12/12/2022

Variational formulation of Lagrange eqs.

There is a different deduction of Lagrange eqs., involving a condition of stationarisation (min, max, saddle...) of a certain "functional". The situation is similar to:

Fermat principle in optics - the path taken between two given points by a ray of light is the path corresponding to minimal time.

Riemann's geometry: a geodesic is the curve of minimal length among all curves joining two given points.

"minimization of a certain quantity"

Calculus of Variations = Differential Calculus for functionals.

\[
\mathcal{J} : \{ \text{space of curves} \} \rightarrow \mathbb{R}
\]

functional, def. in a set of infinite dimension

EXAMPLES

\[
\Gamma_{t_0, t_1} := \left\{ \sigma \in C^0 \left( [t_0, t_1], \mathbb{R}^n \right) \right\}
\]

\[
\mathcal{J}[\sigma] = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \sigma(t) \, dt \quad \text{(average)}
\]

\[
\mathcal{J}[\sigma] = \left( \int_{t_0}^{t_1} \sigma^2(t) \, dt \right)^{1/2} \quad \mathcal{J}[\sigma] = \max_{t \in [t_0, t_1]} |\sigma(t)| \quad \text{(sup norm)}
\]

\[
\mathcal{J}[\sigma] = \sigma(\bar{t}) \quad \text{or} \quad \mathcal{J}[\sigma] = \sigma^1(\bar{t})
\]

(\bar{t} \in [t_0, t_1])

\[
\Gamma_{t_0, t_1} := \left\{ \sigma \in C^0 \left( [t_0, t_1], \mathbb{R}^n \right) \right\}
\]

\[
\mathcal{J}[\sigma] = \int_{t_0}^{t_1} \left\| \frac{d\sigma}{dt}(t) \right\| \, dt = \int_{t_0}^{t_1} \sqrt{\sigma_2^2(t) + \sigma_3^2(t) + \ldots + \sigma_n^2(t)} \, dt \quad \text{(length)}
\]
If the curve is on $\mathbb{R}^2$ (on the plane) and is given by $y = u(x)$, $a \leq x \leq b$, then the length functional becomes:

$$
\int_a^b \sqrt{1 + u'(x)^2} \, dx
$$

$\gamma(x) = (x, u(x))$

**General formulation**

More generally, we are interested in functionals given by a function $U \subseteq \mathbb{R}^n$

$$
L : U \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}
$$

$$(q, \dot{q}, t) \rightarrow L(q, \dot{q}, t)$$

$U$ is open

$L$ is called Lagrangian

$$(\mathcal{M}, \mathfrak{m}_\mathcal{M}) \rightarrow \mathcal{L}(\mathcal{M}, T\mathcal{M})$$

$L : \mathcal{L}(\mathcal{M}, T\mathcal{M}) \rightarrow \mathbb{R}$

$$
\mathcal{J}_L[\gamma] = \int_{\gamma_0}^{\gamma_1} L(\gamma(t), \dot{\gamma}(t), t) \, dt
$$

Action functional associated to

the Lagrangian $L$.

**Important def:** the domain of $\mathcal{J}_L$.

$$
\Gamma = \Gamma_{q_0, q_2} := \{ \gamma \in C^0([t_0, t_2], \mathcal{U}) \mid \begin{cases} 
\gamma(t_0) = q_0 \\
\gamma(t_2) = q_2
\end{cases} \}
$$

**Good question:** what does it mean that $\gamma \in \Gamma$ is a critical curve of $\mathcal{J}_L$??
We define an analogous of the directional derivative for functions \( f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \).

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**Case of Functions**

\( f \in C^1(U, \mathbb{R}) \).

We recall that the directional derivative in a point \( x \in U \) along the direction given by the vector \( \mathbf{v} \in \mathbb{R}^n \) is:

\[
\frac{d}{d\lambda} f(x + \lambda \mathbf{v}) \bigg|_{\lambda = 0} = Df(x) \mathbf{v}
\]

**Analysis II**

Therefore, \( Df(x) = 0 \) (\( x \) is a critical (stationary) point for \( f \)).

\[
\frac{d}{d\lambda} f(x + \lambda \mathbf{v}) \bigg|_{\lambda = 0} = 0 \quad \forall \mathbf{v} \in \mathbb{R}^n
\]

---

**Case of Functionals**

We briefly need to introduce:

\[
\Gamma_0 = \Gamma_{t_0, t_1}^{0, 0} = \{ \gamma \in C^0([t_0, t_1], U) \text{ such that } \gamma(t_0) = 0, \gamma(t_1) = 0 \}
\]

Loops

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**Remarks**

1. \( \Gamma \) is not linearly closed, that is

\[
\forall \gamma, \gamma' \in \Gamma \quad [a \gamma + b \gamma' \notin \Gamma \quad \text{if } \gamma, \gamma' \in \Gamma]
\]

\[
a \gamma(t_0) + b \gamma'(t_0) = a q_0 + b q_0 = q_0
\]

But

\[
\neq q_0
\]
$\Gamma_0 = \Gamma_{t_0, t_1}$ is linearly closed!

\[ a \delta(t_0) + b \delta'(t_0) = 0 \Rightarrow 0 = 0 \]
\[ a \delta(t_2) + b \delta'(t_2) = 0 \]

\[ \Gamma_0 \text{ has a structure of } \mathbb{R} - \text{space}. \]

2) Since $U$ is open ($U \subseteq \mathbb{R}^n$), for every $\delta \in \Gamma$ and $\delta \in \Gamma_0$, we have that:

\[ \delta(t) + \lambda \eta(t) \in U \quad \forall \lambda \in [-\delta, \delta] \quad \forall t \in [t_0, t_2] \]

Moreover

\[ \delta + \lambda \eta \in \Gamma. \]

**DEFINITION**

(iii) A functional $J : \Gamma \to \mathbb{R}$ is Gateaux-differentiable at $\delta \in \Gamma$ if, for every $\eta \in \Gamma_0$, the derivative:

\[ \frac{d}{d\lambda} J[\delta + \lambda \eta] \big|_{\lambda = 0} \]

exists in $\mathbb{R}$. In such a case, we write
Let \( J : \Gamma \to \mathbb{R} \) Gâteaux-differentiable. If \( \gamma \in \Gamma \) is a critical curve of \( J \) if
\[
\frac{d}{d\lambda} J[\gamma + \lambda \xi] \bigg|_{\lambda=0} = 0 \quad \text{for all } \lambda > 0.
\]
Equivalently,
\[
\delta J(\gamma, \xi) = 0 \quad \text{for all } \xi \in \mathbb{R}.
\]

**Proposition**

Let \( L : U \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), be a function, \( U \subseteq \mathbb{R}^n \).

Then

- \( J_L : \Gamma \to \mathbb{R} \) defined in (1) is Gâteaux-differentiable at every \( \gamma \in \Gamma \).

Moreover, for every \( \xi \in \Gamma_0 \), we have:

\[
\delta J_L(\gamma, \xi) = \sum_{i=1}^{m} \int_{t_0}^{t_1} \mathcal{P}_i(t) \left[ \frac{d}{dt} \frac{\partial L}{\partial q_i}(\gamma(t), \dot{\gamma}(t), t) \right] dt.
\]
\[
- \frac{d}{\partial \gamma_i} \left( \vec{g} (\vec{g}(t), \dot{\vec{g}}(t), t) \right) \, dt
\]

**Proof**

It is a "simple" calculation.

Remind that

\[
J_L[\vec{g}] = \int_{t_0}^{t_1} L(\vec{g}(t), \dot{\vec{g}}(t), t) \, dt
\]

we need to write \(J_L[\vec{g} + \lambda \vec{\eta}]\)

\[
J_L[\vec{g} + \lambda \vec{\eta}] = \int_{t_0}^{t_1} L(\vec{g}(t) + \lambda \vec{\eta}(t), \dot{\vec{g}}(t) + \lambda \dot{\vec{\eta}}(t), t) \, dt.
\]

\[
\frac{d}{d\lambda} J_L[\vec{g} + \lambda \vec{\eta}] \bigg|_{\lambda=0} =
\]

\[
= \int_{t_0}^{t_1} \frac{d}{d\lambda} \left[ L(\vec{g} + \lambda \vec{\eta}, \dot{\vec{g}} + \lambda \dot{\vec{\eta}}, t) \right] \, dt \bigg|_{\lambda=0} =
\]

\[
= \int_{t_0}^{t_1} \sum_{i=1}^{n} \left[ \frac{\partial L}{\partial \gamma_i} (\vec{g}(t), \dot{\vec{g}}(t), t) \vec{\eta}_i + \frac{\partial L}{\partial \dot{\gamma}_i} (\vec{g}(t), \dot{\vec{g}}(t), t) \vec{\dot{\eta}}_i \right] \, dt =
\]

\[
= \int_{t_0}^{t_1} \sum_{i=1}^{n} \left[ \frac{\partial L}{\partial \gamma_i} (\vec{g}(t), \dot{\vec{g}}(t), t) \vec{\eta}_i + \left( \frac{d}{dt} \left[ \vec{\eta}_i \frac{\partial L}{\partial \dot{\gamma}_i} \right] \right) \right] \, dt =
\]

by parts

\[
- \vec{\dot{\eta}}_i \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\gamma}_i} \right] \, dt =
\]
\[
\sum_{i=1}^{m} m_i \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i} \right] dt = 0
\]

\[
\sum_{i=1}^{m} m_i \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i} \right] dt = 0
\]

\[
- \sum_{i=1}^{m} m_i \left[ \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_i} \right] dt = 0
\]

\[
\sum_{i=1}^{m} m_i \dot{q}_i \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right] dt = 0
\]

**Theorem:** (Hamilton Principle of Least Action).

\[ L \in C^1(U \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \quad U \subset \mathbb{R}^n \text{ open.} \]

\[ \gamma \in M \text{ is a critical curve of } J_L \text{ if and only if} \]

\[ \gamma \in M \text{ is a solution of Lagrange eqs.} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \forall i = 1 \ldots m. \]

**Proof**

\[ (\Rightarrow) \text{ See previous proposition.} \]

\[ (\Leftarrow) \text{ Central role of } \forall \gamma \in \Pi_0. \]

Suppose that the \( \gamma(t) \) is a critical curve of \( J_L \).
Let write
\[ f_i(t) := \frac{\partial^2}{\partial t^2} \left[ \tilde{a}_i \left( \sigma, \delta, t \right) \right] - \frac{\partial}{\partial q_i} \left( \sigma, \delta, t \right) \]

Since \( \sigma \) is a critical curve for \( J_\lambda \), by previous proposition, this means that
\[
\sum_{i=1}^{n} \int_{t_0}^{t_1} f_i(t) \eta_i(t) \, dt = 0 \quad \forall \eta \in \pi.
\]

We need to prove that \( f_i(t) \equiv 0 \)
\[ \forall \eta \in \pi. \]

We prove that \( f_1(t) \equiv 0 \). The argument for other \( \eta \) is exactly the same!

Let
\[ \pi = (\pi_1, \ldots, \pi_m) \in \pi_0 \]
such that \( \pi_2 = \cdots = \pi_m = 0 \). Then:
\[
\sum_{i=1}^{m} \int_{t_0}^{t_1} f_i(t) \eta_i(t) \, dt = 0 \quad \Rightarrow \quad \text{Only } \pi_1 \neq 0 \]
\[
\int_{t_0}^{t_1} f_1(t) \eta_1(t) \, dt = 0
\]
\[ \forall \eta_1 : [t_0, t_1] \to \mathbb{R} \quad \text{such that } \eta_1(t_0) = \eta_1(t_1) = 0. \]
By contrad., suppose that $F_1 \neq 0$

This means that $\exists \bar{t} \in (t_0, t_1)$ such that $F_2(\bar{t}) \neq 0$. For example, suppose $F_2(F) > 0$.

$\Rightarrow \exists (a, b) \in (t_0, t_1), \bar{t} \in (a, b)$ such that $F_2(t) > 0 \forall t \in (a, b)$.  

\[
\int_{a}^{b} \frac{F_2(t) \mathcal{M}_2(t)}{F_1(t)} dt > 0
\]

$\Rightarrow \int_{a}^{b} \frac{F_2(t) \mathcal{M}_2(t)}{F_1(t)} dt > 0$  

$\Rightarrow F_1(t) \equiv 0$, that is

Choose $\mathcal{M}_2(t)$ as follows:

$\mathcal{M}_2(t) = \begin{cases} 
0 & t < a \\
> 0 & t \in (a, b) \\
0 & t > b 
\end{cases}$

$\Rightarrow \int_{a}^{b} \frac{F_2(t) \mathcal{M}_2(t)}{F_1(t)} dt > 0$
\[ \frac{d}{dt} \alpha - \alpha = 0 \]
\[ \alpha = \text{solution of } L \text{eqs.} \]

Which trajectory follows a system?

Determinism

Finalism

Netwon (1660)

The one determined from initial conditions!

Euler - Lagrange (1700)

The one which minimizes the action!

(The Nature has a scope...)

- First Variational principle: Maupertuis ... 1749
- First Formulation of Calculus of Variations: Euler (1707 - 1783)
  Lagrange (1736 - 1813)
$\downarrow$

Euler - Lagrange