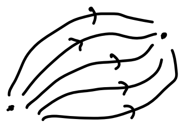
Lesson 32 - 12/12/2022

Variational formulation of Lagrange eqs.

There is a different deduction of Lagrange eqs, involving a condition of stationarization (min, max, saddle ...) of a certain "functional". The situation is similar to:

Fermat principle in Optics → the path taken between two given points by a ray of light is the path corresponding to minimal time.



Riemannian geometry: a geodesic is the curve of minimal length among all curves joining two given points.

$$\Downarrow$$

"minimization of a certain quantity"

$$\Downarrow$$

Calculus of Variations = Differential Calculus for functionals.

$$J : \{\text{space of curves}\} \longrightarrow \mathbb{R}$$

|| functional, def. in a set of infinite dimension

EXAMPLES $\Gamma_{t_0,t_1} := \{\gamma \in \mathcal{C}^\infty([t_0,t_1],\mathbb{R})\}$

- $J[\gamma] = \frac{1}{t_1-t_0}\int_{t_0}^{t_1} \gamma(t)\,dt$ (average) $t_0 \neq t_1$
- $J[\gamma] = \left(\int_{t_0}^{t_1} \gamma^2(t)\,dt\right)^{1/2}$ (euclidean norm) $\qquad J[\gamma] = \max_{t\in[t_0,t_1]} |\gamma(t)|$ (sup norm)
- $J[\gamma] = \gamma(\bar{t})$ $(\bar{t} \in [t_0,t_1])$ or $J[\gamma] = \gamma'(\bar{t})$ $(\bar{t} \in [t_0,t_1])$

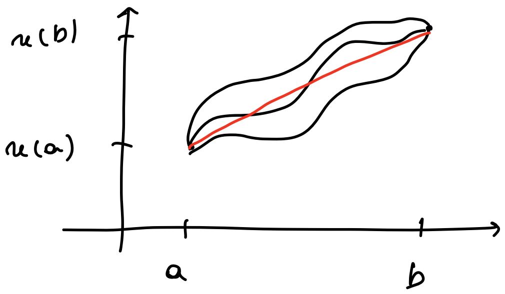
$\Gamma_{t_0,t_1} := \{\gamma \in \mathcal{C}^\infty([t_0,t_1],\mathbb{R}^n)\}$

- $J[\gamma] = \int_{t_0}^{t_1} \left\| \frac{d\gamma}{dt}(t) \right\| dt = \int_{t_0}^{t_1} \sqrt{\dot{\gamma}_1^2(t) + \dot{\gamma}_2^2(t) + \dots + \dot{\gamma}_n^2(t)}\, dt$ (lenght)

If the curve is on $\mathbb{R}^2$ (on the plane) and is given by $y = u(x)$, $a \le x \le b$, then the length-functional becomes:

$$J[\gamma] = \int_a^b \sqrt{1 + u'^2(x)}\, dx$$

$\gamma(x) = (x, u(x))$



General formulation

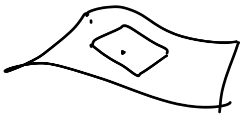
More generally, we are interested on functionals given by a function

$$L : U \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$
$$(q, \dot{q}, t) \mapsto L(q, \dot{q}, t)$$

$U \subseteq \mathbb{R}^n$ open

L is called Lagrangian

(M mnf $\quad TM = \bigcup_{x \in M} T_x M$



$L : TM \times \mathbb{R} \to \mathbb{R}$)

$$\boxed{\begin{array}{c} J_L[\gamma] = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t), t)\, dt \\ \downarrow \end{array}} \qquad (*)$$

Action functional associated to the Lagrangian L.

Important def: the domain of J_L.

$$\Gamma = \Gamma^{q_0, q_1}_{t_0, t_1} := \left\{ \gamma \in \mathcal{C}^{\infty}([t_0, t_1], U) \text{ such that } \begin{cases} \gamma(t_0) = q_0 \\ \gamma(t_1) = q_1 \end{cases} \right\}$$

Crucial question: what does it mean that $\gamma \in \Gamma$ is a CRITICAL CURVE of J_L??

$$\Downarrow$$

We define an analogous of the DIRECTIONAL DERIVATIVE for functions $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$.

→ The case of functions.

$f \in C^1(U, \mathbb{R})$. $\quad (U \subseteq \mathbb{R}^n)$

We recall that the directional derivative in a point $x \in U$ along the direction given by the vector $v \in \mathbb{R}^n$ is:

$$\frac{d}{d\lambda} f(x + \lambda v)\Big|_{\lambda=0} \underset{\substack{\downarrow \\ \text{Analysis II}}}{=} \nabla f(x)\, v$$

Therefore $\nabla f(x) = 0$ (x is a critical / stationary point for f) $\underline{\text{iff}}$

$$\frac{d}{d\lambda} f(x + \lambda v)\Big|_{\lambda=0} = 0 \qquad \forall\, v \in \mathbb{R}^n$$

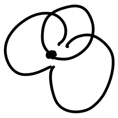
→ The case of functionals

we firstly need to introduce:

$$\Gamma_0 = \Gamma^{0,0}_{t_0,t_1} := \left\{ \eta \in C^\infty([t_0, t_1], U) \text{ such that } \begin{cases} \eta(t_0) = 0 \\ \eta(t_1) = 0 \end{cases} \right\}$$

↓

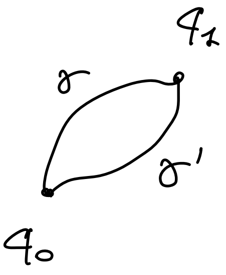
LOOPS



Remarks

1) $\Gamma = \Gamma^{q_0, q_1}_{t_0, t_1}$ is not linearly closed, that is

$a\gamma + b\gamma' \notin \Gamma$ if $\gamma, \gamma' \in \Gamma$ $(a, b \in \mathbb{R})$



$$a\gamma(t_0) + b\gamma'(t_0) = a q_0 + b q_0 \neq q_0$$

But

$\Gamma_0 = \Gamma_{t_0,t_1}^{0,0}$ is linearly closed !

$$a\underbrace{\gamma(t_0)}_{=0} + b\underbrace{\gamma'(t_0)}_{=0} = 0 \Longrightarrow$$

$$a\gamma(t_1) + b\gamma'(t_1) = 0$$

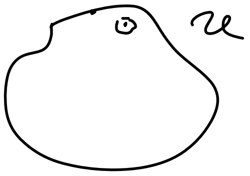
Γ_0 has a structure of $\mathbb{R}$-space.

2) Since $\mathcal{U}$ is open ($\mathcal{U} \subseteq \mathbb{R}^n$), for every $\gamma \in \Gamma$ and $\eta \in \Gamma_0$, we have that

$$\gamma(t) + \lambda\eta(t) \in \mathcal{U} \qquad \forall \lambda \in [-\delta, \delta] \quad \forall t \in [t_0, t_1]$$

Moreover

$$\gamma + \lambda\eta \in \Gamma.$$



DEFINITION

(i) A functional $J : \Gamma \longrightarrow \mathbb{R}$ is Gateax - differentiable at $\gamma \in \Gamma$ if, $\underline{\forall \eta \in \Gamma_0}$, the derivative :

$$\frac{d}{d\lambda} J[\gamma + \lambda\eta] \Big|_{\lambda=0}$$

exists in $\mathbb{R}$. In such a case, we write

$$\delta J(\gamma,\eta) := \frac{d}{d\lambda} J[\gamma+\lambda\eta]\Big|_{\lambda=0}$$

(ii) Let $J:\Gamma\to\mathbb{R}$ Gateaux-differentiable. $\gamma\in\Gamma$ is a critical curve of J if

$$\frac{d}{d\lambda} J[\gamma+\lambda\eta]\Big|_{\lambda=0} = 0 \qquad \forall \eta\in\Gamma_0$$

Equivalently:

$$\delta J(\gamma,\eta) = 0 \qquad \forall \eta\in\Gamma_0$$

Proposition

$L: U\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$, C^1 Lagrangian function, $U\subseteq\mathbb{R}^n$.

Then

- $J_L:\Gamma\to\mathbb{R}$ defined in $(*)$ is Gateaux-differentiable at every $\gamma\in\Gamma$.
- Moreover, $\forall\eta\in\Gamma_0$, we have:

$$\delta J_L(\gamma,\eta) =$$

$$= -\sum_{i=1}^{m}\int_{t_0}^{t_1}\eta_i(t)\Big[\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot q_i}(\gamma(t),\dot\gamma(t),t)\Big) -$$

$$- \frac{\partial \mathcal{L}}{\partial q_i}(\gamma(t), \dot\gamma(t), t))\Big] \, dt$$

Proof It is a "simple" calculation.

Remind that

$$J_L[\gamma] = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t), t)\, dt$$

We need to write $J_L[\gamma+\lambda\eta]$

$$J_L[\gamma+\lambda\eta] = \int_{t_0}^{t_1} L(\gamma(t)+\lambda\eta(t), \dot\gamma(t)+\lambda\dot\eta(t), t)\, dt.$$

$$\frac{d}{d\lambda} J_L[\gamma+\lambda\eta]\Big|_{\lambda=0} =$$

$$= \int_{t_0}^{t_1} \frac{d}{d\lambda}\left[L(\gamma+\lambda\eta, \dot\gamma+\lambda\dot\eta, t)\right]dt \,\Big|_{\lambda=0} =$$

$$= \int_{t_0}^{t_1}\sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i}(\gamma,\dot\gamma,t)\,\eta_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot q_i}(\gamma,\dot\gamma,t)\,\dot\eta_i}\right] dt =$$

$$\overset{\text{by parts}}{=} \int_{t_0}^{t_1}\sum_{i=1}^{n} \left[\underbrace{\frac{\partial \mathcal{L}}{\partial q_i}(\gamma,\dot\gamma,t)\,\eta_i} + \boxed{\frac{d}{dt}\left[\eta_i \frac{\partial \mathcal{L}}{\partial \dot q_i}\right]} - \underbrace{\eta_i \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i}}\right] dt =$$

$$= \underbrace{\sum_{i=1}^{n} \eta_i \frac{\partial L}{\partial \dot q_i} \Big|_{t_0}^{t_1}}_{\equiv 0} + \qquad \underbrace{\text{since } \eta_i(t_1) = \eta_i(t_0) = 0 \ \forall i = 1 \ldots n}$$

$$+ \sum_{i=1}^{n} \int_{t_0}^{t_1} \eta_i \left[\underbrace{\frac{\partial L}{\partial q_i}} - \underbrace{\frac{d}{dt} \frac{\partial L}{\partial \dot q_i}} \right] dt =$$

$$= - \sum_{i=1}^{n} \int_{t_0}^{t_1} \eta_i \left[\frac{d}{dt} \frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} \right] dt \qquad \square$$

THEREOM (Hamilton Principle of Least Action).

It is a sort of widely accepted "thriftiness" of Nature in all its actions.

$L \in C^1(U \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $U \subseteq \mathbb{R}^n$ open.

$\gamma \in \Gamma$ is a critical curve of J_L

iff

$\gamma \in \Gamma$ is a solution of Lagrange eqs.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \forall i = 1 \ldots n.$$

PROOF ($\Leftarrow$) See previous proposition.

($\Rightarrow$) Central role of: $\forall \eta \in \Gamma_0$!!

Suppose that $t \mapsto \gamma(t)$ is a critical curve of J_L

Let write

$$f_i(t) := \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}(\gamma, \dot{\gamma}, t)\right] - \frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}, t)$$

Since γ is a critical curve for J_L, by previous proposition, this means that

$$\sum_{i=1}^{n} \int_{t_0}^{t_1} f_i(t)\, \eta_i(t) = 0 \qquad \forall\, \eta \in \Gamma_0 .$$

We need to prove that $f_i(t) \equiv 0$ $\forall\, i = 1 \dots n$.

We prove that $f_1(t) \equiv 0$. The argument for other eqs. is exactly the same!

Let

$\eta = (\eta_1 \dots \eta_n) \in \Gamma_0$

such that $\eta_2 = \dots = \eta_n = 0$ Then:

$$\sum_{i=1}^{n} \int_{t_0}^{t_1} f_i(t)\, \eta_i(t) = 0 \quad \underset{\text{Only } \eta_1 \neq 0}{\iff}$$

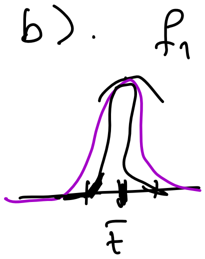
$$\int_{t_0}^{t_1} \underline{f_1}(t)\, \eta_1(t) = 0$$

$\forall\, \eta_1 : [t_0, t_1] \longrightarrow \mathbb{R}$ such that $\eta_1(t_0) = \eta_1(t_1) = 0$

By contradd., suppose that $f_1 \neq 0$

This means that $\exists\ \bar{t} \in (t_0, t_1)$ such that $f_1(\bar{t}) \neq 0$, for example suppose $f_1(\bar{t}) > 0$.

$\Rightarrow \exists\ (a, b) \subseteq (t_0, t_1)$, $\bar{t} \in (a, b)$ such that $f_1(t) > 0 \quad \forall\ t \in (a, b)$.



$$\Downarrow$$

Choose $\eta_1(t)$ as follows:

$$\eta_1(t) = \begin{cases} 0 & t < a \\ > 0 & t \in (a, b) \\ 0 & t > b \end{cases}$$

$$\int_{t_0}^{t_1} \underline{f_1(t)}\, \eta_1(t) =$$

$$= \int_a^b \underbrace{f_1(t)}_{>0} \underbrace{\eta_1(t)}_{>0} dt > 0 \quad \lightning$$

$\Rightarrow f_1(t) \equiv 0$!! that is
?.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} = 0$$

$\Rightarrow \gamma$ is a solution of $\mathcal{L}$ eqs. $\square$

Which trajectory follows a system

DETERMINISM ↙ ↘ FINALISM

Newton 1600

The one determined from initial conditions!

Euler-Lagrange 1700

The one which minimizes the Action!

(The Nature has a scope...)

- First Variational principle: Maupertuis ... 1744
- First formulation of Calculus of Variations Euler (1707-1783) Lagrange (1736-1813)

↓

Euler-Lagrange eqs.