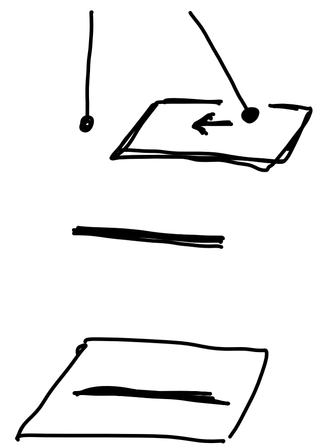
Lesson 31 - 07/12/2022

- Foucault pendulum
- Ex.

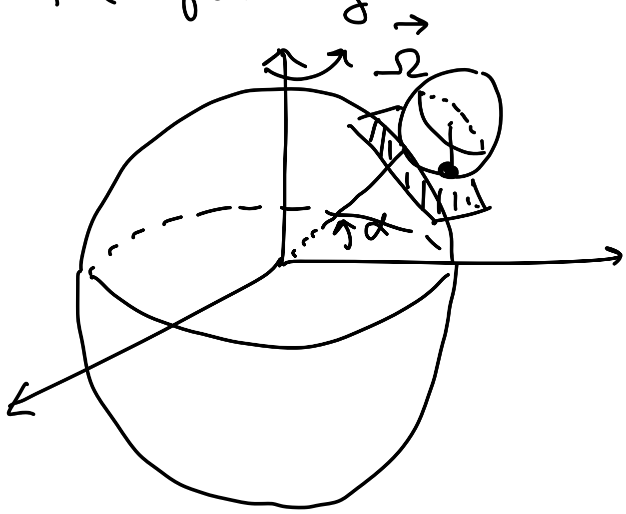
J.B. Léon Foucault → 1851 Panthéon Paris
with a pendulum of lenght 67 mt., 28 kg.



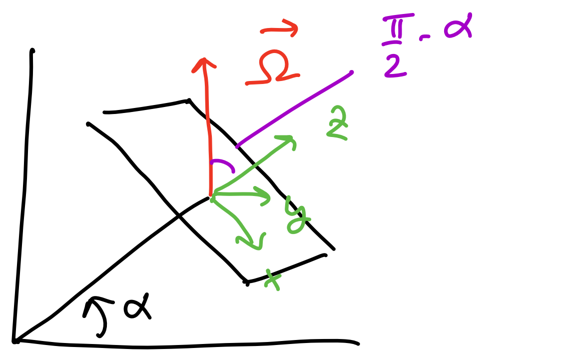
— x —

Model P, m

constrained on a sphere of radius R and subj. to the gravity.



Coordinates x, y = projections of the local tg plane



$$\vec{\Omega} = (\underbrace{-\Omega\cos\alpha}_{\Omega_x}, \underbrace{0}_{\Omega_y}, \underbrace{\Omega\sin\alpha}_{\Omega_z})$$

Facts • The centrifugal acc. is negligible w.r.t. to gravity.

centr. acc $\simeq 3 \cdot 10^{-3}$ grav. acc.

• Moreover, since we are in a rotating system, we need to consider the Coriolis force:

$$\vec{F}_{cor} = -2m\,\vec{\Omega} \wedge \vec{v}.$$

• We study approx. eqs. near the south pole, which is a stable eq. $\Rightarrow$ the linearized eqs. near this equilibrium give a good approx. of the problem.

$$\Downarrow$$

We take into account the linearized eqs. of the spherical pendulum near the south pole.

$$\begin{cases} m\ddot{x} = -\dfrac{mg}{R}x + \dots \\ m\ddot{y} = -\dfrac{mg}{R}y + \dots \end{cases}$$

linear terms of the Lagr. comp. of the Coriolis force.

In the next calculations, we write the liner terms of the Lagr. components for the Coriolis force.

$$dL = Q_x(x, y, \dot{x}, \dot{y})\,dx +$$

$$+ Q_y(x, y, \dot{x}, \dot{y})\,dy =$$

$$= -2m\vec{\Omega} \wedge \vec{v} \cdot d\vec{OP}$$

$$(x, y) \longmapsto (x, y, \underbrace{-\sqrt{R^2 - x^2 - y^2}}_{z(x,y)})$$

parameterization of the south emisphere where we are studying our problem.

$$d\vec{OP} = (dx, dy, dz(x, y)) =$$

$$= \left(dx, dy, \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$$

$$\vec{v} = (\dot{x}, \dot{y}, \dot{z}) =$$

$$= \left(\dot{x}, \dot{y}, \frac{\partial z}{\partial x}\dot{x} + \frac{\partial z}{\partial y}\dot{y} \right)$$

$\Downarrow$

$$dL = -2m\, \vec{\Omega} \wedge \vec{v} \cdot d\vec{OP} =$$

$$= \det \begin{pmatrix} dx & dy & dz \\ -2m\Omega_x & 0 & -2m\Omega_z \\ \dot{x} & \dot{y} & \dot{z} \end{pmatrix} =$$

$$= (2m\Omega_z \dot{y})\, dx +$$

$$+ (2m\Omega_x \dot{z} - 2m\Omega_z \dot{x})\, dy +$$

$$+ (-2m\Omega_x \dot{y})\, dz(x,y) =$$

$$= (2m\Omega_z \dot{y})\, dx +$$

$$\left[2m\Omega_x \left(\frac{\partial z}{\partial x}\dot{x} + \frac{\partial z}{\partial y}\dot{y} \right) - 2m\Omega_z \dot{x} \right] dy$$

$$+ (-2m\Omega_x \dot{y}) \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right)$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{R^2-x^2-y^2}}$$

If I consider only linear terms:

$$dL = [2m\Omega_z \dot{y} + \mathcal{O}(|(x, y, \dot{x}, \dot{y})|)]dx$$
$$+ [-2m\Omega_z \dot{x} + \mathcal{O}(|(x, y, \dot{x}, \dot{y})|)]dy$$

ω

$$\begin{cases} \cancel{m}\ddot{x} = -\frac{\cancel{m}g}{R} x + 2\cancel{m}\Omega_z \dot{y} \\ \cancel{m}\ddot{y} = -\frac{\cancel{m}g}{R} y - 2\cancel{m}\Omega_z \dot{x} \end{cases}$$

$$\omega^2 = g/R$$

$$\begin{cases} \ddot{x} = -\omega^2 x + 2\Omega_z \dot{y} \\ \ddot{y} = -\omega^2 y - 2\Omega_z \dot{x} \end{cases}$$

Analytical study

$$\begin{cases} \ddot{x} = -\omega^2 x + 2\Omega_z \dot{y} \\ i\ddot{y} = -\omega^2 i y - 2\Omega_z i \dot{x} \end{cases}$$

$\longrightarrow$ SUM

$$\ddot{x} + i\ddot{y} =$$

$$= \boxed{-\omega^2 x - \omega^2 i y} + 2\Omega_z \dot{y} -$$

$$- 2\Omega_z i \dot{x}$$

$$\xi = x + iy \in \mathbb{C}$$

$$\ddot{\xi} = -\omega^2 \xi - 2i\Omega_z \dot{\xi}$$

$$\downarrow$$

$$\begin{cases} \ddot{\xi} + 2i\Omega_z \dot{\xi} + \omega^2 \xi = 0 \\ \xi(0), \ \dot{\xi}(0) \quad \text{initial data.} \end{cases}$$

$$\downarrow$$

characteristic eq.

$$\lambda^2 + 2i\Omega_z \lambda + \omega^2 = 0$$

$$\lambda_{1,2} = -i\Omega_z \pm \sqrt{\underbrace{-\Omega_z^2 - \omega^2}_{<0}} =$$

$$= i\left(-\Omega_z \pm \sqrt{\Omega_z^2 + \omega^2}\right)$$

General solution:

$$\xi(t, c_1, c_2) =$$

$c_1, c_2 \in \mathbb{C}$ depend on initial conditions.

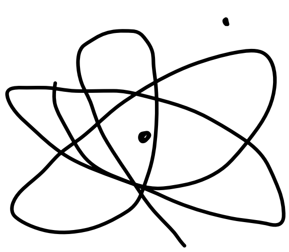
$$= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} =$$

$$= \underbrace{e^{-i\Omega_z t}}_{\text{independent on initial data}} \underbrace{\left(c_1 e^{i\sqrt{\Omega_z^2+\omega^2}\,t} + c_2 e^{-i\sqrt{\Omega_z^2+\omega^2}\,t} \right)}_{\text{represents an ellipse on the plane } Oxy}$$

$\downarrow$

and represents a rotation with angular velocity $-\Omega_z < 0$

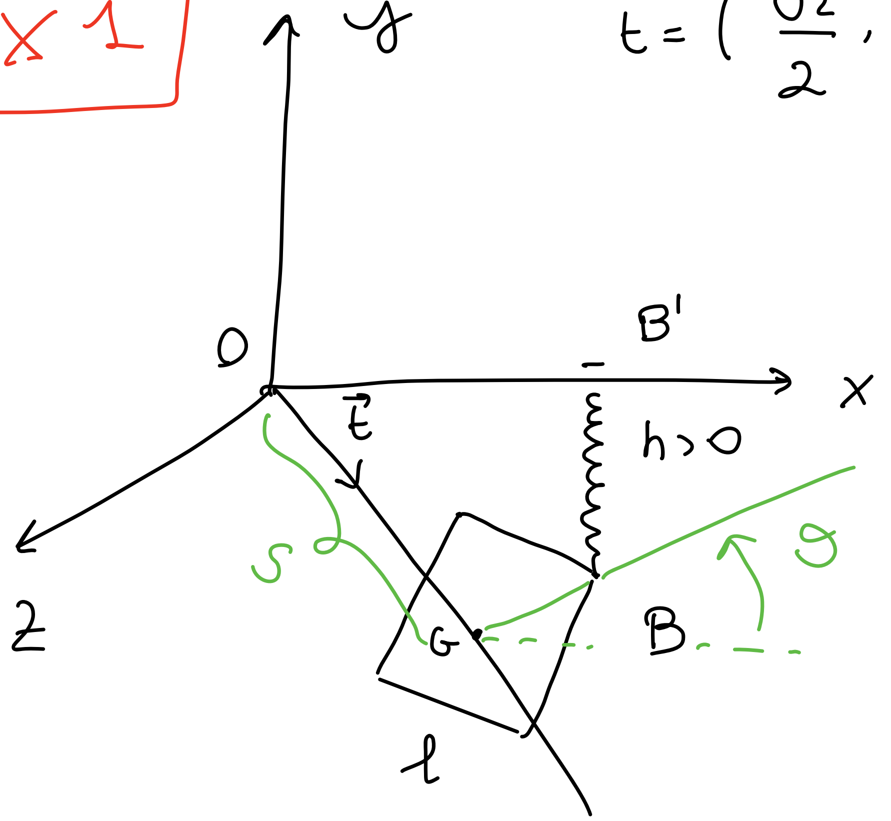
$\Rightarrow$ clockwise!



EX 1

$\vec{t} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right)$

$\downarrow \vec{g}$



- Eq.
- Stab.
- Kinetic energy.
- what about the stability of the stable eq. if we add $\vec{F} = -K \vec{v}_G$, $K > 0$?

Potential

$$U(s,\theta) = U_{gr} + U_{el} =$$

$$= mg\,y_G + \frac{h}{2} y_B^2 =$$

$$= -mg\,s\frac{\sqrt{2}}{2} + \frac{h}{2}\Big(-\frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}\ell \sin\theta\Big)^2 =$$

semi-diagonal of the square.

$$= -mg\,s\frac{\sqrt{2}}{2} + \frac{h}{2}\Big(\frac{\sqrt{2}}{2}(\ell\sin\theta - s)\Big)^2$$

$$\begin{cases} U_\theta = \dfrac{h}{2}\boxed{(\ell\sin\theta - s)}\,\ell\,\boxed{\cos\theta} = 0 \\[2ex] U_s = -mg\dfrac{\sqrt{2}}{2} - \dfrac{h}{2}(\ell\sin\theta - s) = 0 \end{cases}$$

FIRST CASE

$$\vartheta = \pi/2 : \quad -mg\frac{\sqrt{2}}{\cancel{2}} - \frac{h}{\cancel{2}}(\ell - s) = 0$$

$$s = \ell + \frac{mg\sqrt{2}}{h}$$

$$EQ_1 = \left(\pi/2 , \ \ell + \frac{mg\sqrt{2}}{h} \right)$$

$$\vartheta = \frac{3\pi}{2} : \quad -mg\sqrt{2} - h(-\ell - s) = 0$$

$$s = -\ell + \frac{mg\sqrt{2}}{h}$$

$$EQ_2 = \left(\frac{3}{2}\pi , \ -\ell + \frac{mg\sqrt{2}}{h} \right)$$

SECOND CASE

$\ell \sin\theta = s \Rightarrow -mg\frac{\sqrt{2}}{2} = 0$ ↯

$\exists$ only 2 equilibria

$(EQ_1, 0, 0)$ $(EQ_2, 0, 0)$.

<u>Stability</u>

$$\mathrm{Hess}\,U(s,\theta) = \begin{pmatrix} \frac{\partial^2 U}{\partial s^2} & \frac{\partial^2 U}{\partial s \partial \theta} \\ \frac{\partial^2 U}{\partial s \partial \theta} & \frac{\partial^2 U}{\partial \theta^2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}h & -\frac{h}{2}\ell\cos\theta \\ -\frac{h}{2}\ell\cos\theta & \frac{h\ell^2}{2}(\cos^2\theta - \sin^2\theta) + \\ & + \frac{h\ell}{2} s\sin\theta \end{pmatrix}$$

$$\mathrm{Hess}\,\mathcal{U}(EQ_1) = \begin{pmatrix} h/2 & 0 \\ 0 & \ell m g \frac{\sqrt{2}}{2} \end{pmatrix}$$

$\in \mathrm{Sym}^+ \Rightarrow$ STABLE.

$$\mathrm{Hess}\,\mathcal{U}(EQ_2) = \begin{pmatrix} h/2 & 0 \\ 0 & -\ell m g \frac{\sqrt{2}}{2} \end{pmatrix}$$

$\Rightarrow$ UNSTABLE for non-deg. Hessian THEO.

—x—x—

Kinetic energy.

$$K = \frac{m}{2} |\vec{v}_G|^2 + \frac{1}{2} (\vec{\omega}, I_G \vec{\omega})$$

$$= \frac{m}{2}\dot{s}^2 + \frac{1}{2}\left(\frac{ml^2}{6}\right)\dot{\theta}^2$$

$$Q(s,\theta) = \begin{pmatrix} m & 0 \\ 0 & \frac{ml^2}{6} \end{pmatrix}$$

$$Q_s(s,\dot{s}) = -k\dot{s}$$

$$Q_\theta \equiv 0$$

EQ_1 (for L-D theo)

remains stable.

At home

Surface of revolution:

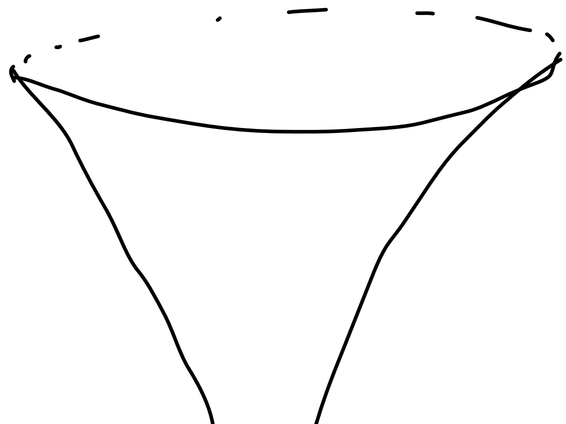
$$z = -R^2 \Big/ \sqrt{x^2+y^2}$$

$R>0$

$\vec{g}$ Parameterization:

$$(s,\theta) \longmapsto \underbrace{\left(Rs\cos\theta,\ Rs\sin\theta,\ -R/s\right)}_{\vec{OP}}$$

$s>0$



$$V_{gr} = -mg\frac{R}{s} \qquad L = K - V$$

L, Routh reduction,
equilibria stability for
the reduced system,
explicit sol. of the original
system corresp. to the equilibrium
of the reduced one.