Spherical pendulum.

Point \( P \), \( m \) contained on a sphere and subjected to the gravity

\[
\begin{align*}
\left\{ \begin{array}{l}
x = R \sin \theta \cos \psi \\
y = R \sin \theta \sin \psi \\
z = R \cos \theta
\end{array} \right. \\
\Rightarrow \left\{ \begin{array}{l}
\dot{x} = R \dot{\theta} \cos \theta \cos \psi - R \dot{\psi} \sin \theta \sin \psi \\
\dot{y} = R \dot{\theta} \cos \theta \sin \psi + R \dot{\psi} \sin \theta \cos \psi \\
\dot{z} = -R \dot{\psi} \sin \theta
\end{array} \right.
\end{align*}
\]

\[
\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2 = \ldots = R^2 \dot{\theta}^2 \cos^2 \theta \cos^2 \psi + R^2 \dot{\psi}^2 \sin^2 \theta \sin^2 \psi + R^2 \dot{\psi}^2 \sin^2 \theta \cos^2 \psi + R^2 \dot{\theta}^2 \sin^2 \theta =
\]

\[
= \frac{R^2 \dot{\theta}^2 \cos^2 \theta + R^2 \dot{\psi}^2 \sin^2 \theta + R^2 \dot{\psi}^2 \sin^2 \theta}{R^2 \dot{\theta}^2 + R^2 \dot{\psi}^2 \sin^2 \theta} =
\]

\[
L = L(\theta, \psi, \dot{\theta}, \dot{\psi}) = \frac{m R^2}{2} \left( \dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2 \right) - mg R \cos \theta
\]

we can divide by \( m R^2 \)

\[
\frac{1}{2} \left( \dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2 \right) - \left( \frac{\dot{\psi}}{R} \right)^2 \cos \theta
\]

\[
L = \frac{1}{2} \left( \dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2 \right) - K \cos \theta
\]

\[
\Rightarrow L(\theta, \dot{\theta}, \dot{\psi}) \quad \text{But } \psi \text{ is a cyclic coord.}
\]

\[
\frac{\partial L}{\partial \dot{\psi}} = J = \sin^2 \theta \dot{\psi} \quad \Rightarrow \dot{\psi} = J / \sin^2 \theta
\]

\[
\frac{\partial L}{\partial \dot{\psi}} (J \neq 0)
\]
Reduced Lagrangian:

\[ L_R(\theta, \dot{\theta}) = \frac{1}{2} \left( \dot{\theta}^2 + \frac{J^2}{\sin^2 \theta} \right) - K \cos \theta - J \left( \frac{\dot{\theta}}{\sin \theta} \right) \]

\[ = \frac{1}{2} \dot{\theta}^2 - \left( \frac{K \cos \theta + \frac{J}{2 \sin \theta}}{2 \sin^2 \theta} \right) V_R(\theta) \]

Lo Lagrangian of a 1-dim conservative dynamical system.

where \( \theta \in (0, \pi) \). We can draw the phase portrait:

\[ \ddot{\theta} = -V_R'(\theta) \quad (J \neq 0) \]

Graph of \( V_R(\theta) = K \cos \theta + \frac{J}{2 \sin^2 \theta} \)

\[ \lim_{\theta \to 0^+} V_R(\theta) = \lim_{\theta \to \pi^-} V_R(\theta) = +\infty. \]

\( \Rightarrow V_R(\theta) \) has at least 1 critical point \( \in (0, \pi) \), which is a minimum. Is it the unique critical point?

we study the \( V_R'(\theta) \).

\[ V_R'(\theta) = -K \sin \theta - \frac{AJ^2 \cos \theta \sin \theta}{\sin^3 \theta} \]

\[ = -K \sin \theta - \frac{J^2 \cos \theta}{\sin^3 \theta} \]

\[ = \frac{-1}{\sin^3 \theta} \left[ K \sin^4 \theta + J^2 \cos \theta \right] \]

\[ \theta \in (0, \pi) \)

\[ V_R'(\theta) = 0 \Rightarrow K \sin^4 \theta = -J^2 \cos \theta \]
For $J \neq 0$, the reduced system has an unique stable equilibrium and all other motions are periodic, with period $T = T(E, J)$.

Reconstruction of the dynamics for the original system.
Recall that \( \varphi = J / \sin^2 \theta \).
Therefore, if \( t \rightarrow \Theta(t) \) is a \((\text{periodic})\) solution of the reduced system, then
\[
\varphi(t) = \varphi(0) + \int_0^t \frac{J}{\sin^2[\Theta(s)]} \, ds
\]

Equilibrium:
\[
\Theta(t) = \Theta_m(J), \quad \varphi(t) = \varphi_0 + \frac{J t}{\sin^2[\Theta_m(J)]}
\]

\( \Rightarrow \) The pendulum rotates with constant angular velocity \( J / \sin^2 \Theta_m(J) \).

Some qualitative properties of other motions.
1) The cord \( \Theta \) follows a periodic motion between a minimal value \( \Theta_- \) and maximal value \( \Theta_+ \).
2) Moreover, \( \dot{\varphi}(t) = J / [\sin^2(\Theta(t))] \rightarrow 0 \) if \( J > 0 \)
\( \Rightarrow \) \( \varphi(t) \) is monotonically \( > 0 \) increasing (or decreasing).
3) Natural question is: The motions of the spherical pendulum are periodic? \((\Leftrightarrow)\) trajectories are closed??
it can be proved that, varying $E$ and $J$, the
motion continues to change from periodic (closed
trajectories) to aperiodic (dense trajectories).

---x---

For previous problem, coordinates $x, y$ are
declined. However, for examples, in order to study
small oscillations of the spherical pendulum
around the stable eq. (south pole) we
need to change coordinates.

\[
\begin{align*}
x & = x \\
y & = \sqrt{R^2 - x^2 - y^2}
\end{align*}
\]

(1) Linearized eqs. of the spherical pendulum
around the south pole.

Linearization around an equilibrium $Br$ a mechanical
Lagrangian is given by Lagrange eq for

\[
L^0(x, y, \dot{x}, \dot{y}) = \frac{1}{2} \Delta (0, 0) \left( \frac{\dot{x}}{\dot{y}} \right)
\]

\[
= \frac{1}{2} \text{Hess} \left( 0, 0 \right) \left( \frac{x}{y} \right) \left( \frac{\dot{y}}{\dot{x}} \right)
\]

\[2 = \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{R^2 - x^2 - y^2}}\]

\[= 0, K = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \left( \frac{x^2 + y^2}{R^2 - x^2 - y^2} \right) \right) = \]

\[= \frac{1}{2} m \left[ \dot{x}^2 \left( 1 + \frac{x^2}{R^2 - x^2 - y^2} \right) + \dot{y}^2 \left( 1 + \frac{y^2}{R^2 - x^2 - y^2} \right) + \frac{2xy \dot{x} \dot{y}}{(R^2 - x^2 - y^2)} \right] \]
\[ V(x, y) = mg^2 = -mg \sqrt{R^2 - x^2 - y^2} \]

\[ L = K - V \]

\[ \mathbf{\Omega}(0, 0) = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ \nabla V(x, y) = mg \left( \frac{x}{(R^2 - x^2 - y^2)^{3/2}}, \frac{y}{(R^2 - x^2 - y^2)^{3/2}} \right) \]

\[ \text{Hess} \ n(x, y) = \frac{mg}{R} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ L^0(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{mg}{2R} (x^2 + y^2) \]

\[
\begin{aligned}
\frac{d}{dt} \left( \frac{\partial L^0}{\partial \dot{x}} \right) - \frac{\partial L^0}{\partial x} &= 0 \\
\frac{d}{dt} \left( \frac{\partial L^0}{\partial \dot{y}} \right) - \frac{\partial L^0}{\partial y} &= 0
\end{aligned}
\]

\[
\begin{aligned}
m \ddot{x} &= -\frac{mg}{R} x \\
m \ddot{y} &= -\frac{mg}{R} y
\end{aligned}
\]

are the linearized eqs. of the spherical pendulum around 
\((0, 0) \to \text{south pole (stable eq.)} \)

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**Potentials depending on velocities:** the magnetic stabilization

- Electric charge on the plane, subjected to a conservative repulsive force:
\[ \vec{F} = -e\vec{B}\vec{v} \]

\[ V(r) = -\frac{1}{2}Kr^2 \]

\((0,0,0,0)\) is an unstable equilibrium \(\forall K \geq 0.\)

\[ L = \frac{1}{2}m(\dot{r}^2 + \dot{\theta}^2) + \frac{1}{2}Kr^2 \]

\(K=0:\) instability of the free particle.

\(K>0:\) instability of the harmonic repeller.

- We add a magnetic field, \(\vec{B} \equiv B\vec{e}_z\)

\[ \Rightarrow \vec{F} = -e\vec{B}\vec{v} = e\vec{B}\vec{v}\wedge \vec{e}_z \]

\[ K=0 \quad \text{→ A charge on a magnetic field.} \]

In such a case, the motion through \((x_0,y_0)\) is circular with radius:

\[ R = \frac{mv_0}{eB} \]

\[ v_0 = \sqrt{x_0^2 + y_0^2} \]

\(\Rightarrow\) Every point \((x_0,y_0,0,0)\) is stable.
The origin is stable if

\[ B^2 > 4 \frac{km}{e^2} \]

New Lagrangian:

\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k z^2 - \frac{1}{2} e B \cdot \dot{z} \]

Now

\[ \frac{1}{2} e B \cdot \dot{z} = \frac{1}{2} e B \dot{z}^2 = \frac{1}{2} m \frac{\dot{z}^2}{m} = \frac{1}{2} \dot{z}^2 \]

\[ \omega = \frac{e B \dot{z}}{2m} \]

"angular velocity"

Generalized potential for a Coriolis force.

In the inertial system:

\[ \hat{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k r^2 - \frac{1}{2} m \omega^2 z^2 - V_0(z) \]
\[ V_0(z) = \frac{1}{2} (m u^2 - k) z \]

Equilibrium \( z = 0 \) (\( z, \theta \) = (0, 0)) which results stable when

\[ m u^2 - k > 0 \]

\[ \Rightarrow \frac{e^2 B^2}{4 m^2} - k > 0 \]

\[ \Rightarrow B^2 > \frac{k \cdot 4 m}{e^2} \]

Remark: This stability is weak, indeed it is destroyed by adding a fiction, because adding e fiction - \( e \) decreases \( \Rightarrow F = e B v \) \( \Rightarrow e \) decreases itself.

- x - x -