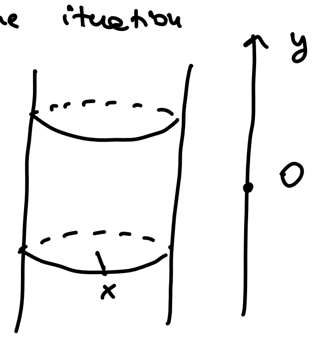
Lesson 29 - 05/12/2022

Maths is not only theorems and solutions: we need also a X-factor!

EX 1 Chaos in "simple" dynamical systems. THE STANDARD MAP.

Discrete dyn. system on the cylinder given by the iteration of the following map:



$$\phi : \begin{cases} \bar{x} = x + y \pmod 1 \\ \bar{y} = y \end{cases}, \quad (x,y) \in S^1 \times \mathbb{R}.$$

y identifies an horizontal section of the cylinder, where the dynamics is given by the translation: $\bar{x} = x + y \pmod 1$.

Every set $S^1 \times \{y\}$ is an invariant set.

Two cases:

1) $y = \frac{n}{m} \in \mathbb{Q}$. Then:

$$x \xmapsto{\phi^1} \underbrace{x + \frac{n}{m} \pmod 1}_{} \xmapsto{\phi^1 \circ \phi^1} \left(x + \frac{n}{m}\right) + \frac{n}{m} \overset{\pmod 1}{\longmapsto} \dots \longmapsto x + \cancel{m} \cdot \frac{n}{\cancel{m}} = x \pmod 1$$

$$\left(x + \frac{n}{m}\right) + \frac{n}{m} = x + \frac{2n}{m}$$

after m-iterations

$\Rightarrow$ the orbit is PERIODIC

2) Let $y \notin \mathbb{Q}$ $(y \in \mathbb{R} \setminus \mathbb{Q}) \Rightarrow$ the orbit is not PERIODIC but it results dense in $S^1 \times \{y\}$.

Natural and crucial question in dyn. systems....

What does it happen if we perturb ϕ?

Do invariant sets persist?

$$\phi_\varepsilon : \begin{cases} \bar{x} = x + y \pmod 1 \\ \bar{y} = y + \varepsilon f(\bar{x}) = y + \varepsilon f(x + y) \end{cases}$$

$f \in C^{\infty}(S^1)$. $\searrow$ STANDARD MAP.

$\varepsilon = 0 : \quad \phi_0 = \phi$.

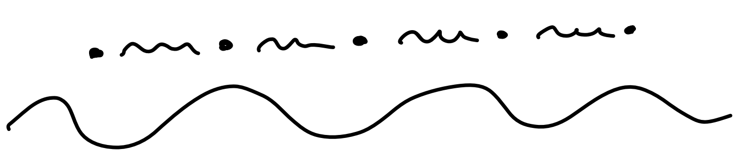
ε very small parameter.

- Genesis of standard map ....

The Frenkel-Kontorova model.

1-dim. infinite chain of particles connected by springs of spring constant $= 1$. Moreover, the system is subj. to a periodic potential:

$$V(x_J) = \frac{K}{4\pi^2} \cos(2\pi x_J) \qquad J \in \mathbb{Z}.$$



Total potential energy of the system:

$$\sum_J \frac{1}{2}(x_J - x_{J-1})^2 + \sum_J V(x_J) =$$

$$= \sum_J \frac{1}{2}(x_J - x_{J-1})^2 + \sum_J \frac{k}{4\pi^2}\cos(2\pi x_J)$$

Equilibria (states):

$$\frac{\partial}{\partial x_J} V = 0 \quad \forall J \in \mathbb{Z} \iff$$

$$\boxed{\underbrace{(x_J - x_{J-1})}_{y_{J-1}} - \underbrace{(x_{J+1} - x_J)}_{y_J} - \frac{k}{2\pi}\sin(2\pi x_J) = 0}$$

Use x-factor! Define: $\boxed{y_J = x_{J+1} - x_J}$

By this def. the previous eq. for equilibria becomes:

$$y_{J-1} - y_J - \frac{k}{2\pi}\sin(2\pi x_J) = 0$$

$$\begin{cases} x_{J+1} = x_J + y_J \pmod 1 \\ y_J = y_{J-1} - \frac{k}{2\pi}\sin(2\pi x_J) \end{cases}$$

$$\begin{cases} x_{J+1} = x_J + y_J \pmod 1 \\ y_{J+1} = y_J - \frac{k}{2\pi} \sin(2\pi x_{J+1}) \end{cases}$$

Equivalently:

$$\begin{cases} \bar{x} = x + y \pmod 1 \\ \bar{y} = y - \frac{k}{2\pi} \sin(2\pi(x+y)) \end{cases}$$

$\longrightarrow$ This is the standard map.
for $f = -\sin(2\pi x)$.

Equilibria states for a physical system $\underline{\Leftrightarrow}$ orbits of a well-known map in dynamical systems.

$\hookrightarrow$ Mathematicians proved by a "variational approach" that $\forall \varepsilon > 0$ the standard map ϕ_ε admits invariant sets.

EX2 A non-trivial ex. of LIMIT CYCLE:
The Van der Pol equation.

$$\ddot{x} + \mu(x)\dot{x} + \omega^2 x = 0 \quad , \quad x \in \mathbb{R}.$$

comes from the analysis of electric circuits. We consider the case:

$$\mu(x) = \mu_0 \left(\frac{x^2}{a^2} - 1 \right) \qquad \mu_0 > 0$$

so that

- $\mu(x) > 0$ (for $x > a$ or $x < -a$) the system lose energy.
- $\mu(x) < 0$ (for $x \in]-a, a[$) the system gain energy.

- Dimensionless form for the previous eq.

→ we use a as unit measure for x
→ we use $\frac{1}{\omega}$ as unit measure for t

$y = x/a \quad , \quad \tau = \omega t$

$$\cancel{a}\,\omega^2 \ddot{y} + \mu_0\left(\frac{\cancel{a^2}\,y^2}{\cancel{a^2}} - 1\right)\cancel{a}\,\omega\,\dot{y} + \omega^2 \cancel{a}\,y = 0$$

$$\ddot{y} + \frac{\mu_0}{\omega}\left(y^2 - 1\right)\dot{y} + y = 0$$

$$\beta = \mu_0/\omega \quad > 0$$

$$\boxed{\ddot{x} + \beta\left(x^2 - 1\right)\dot{x} + x = 0}$$

↳ <u>Van der Pol equation</u>

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\beta\left(x^2 - 1\right)v - x \end{cases}$$

$\exists!$ EQUILIBRIUM $(0,0) \in \mathbb{R}^2$.

<u>PROPOSITION</u> $\forall \beta > 0$ the Van der Pol equation $\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0$

admits a limit cycle with basin of attraction $\mathbb{R}^2 \setminus \{(0,0)\}$.

PROOF We divide the proof into 2 cases.

$\beta \gg 1$

$$\ddot{x} + \beta(x^2-1)\dot{x} + x = 0$$

$$\ddot{x} + \beta(x^2-1)\dot{x} = -x$$

$$\frac{d}{dt}\left[\dot{x} + \beta\left(\frac{x^3}{3} - x\right)\right] = -x$$

we impose

$$\dot{x} + \beta\underbrace{\left(\frac{x^3}{3} - x\right)}_{\gamma(x)} = \beta y$$

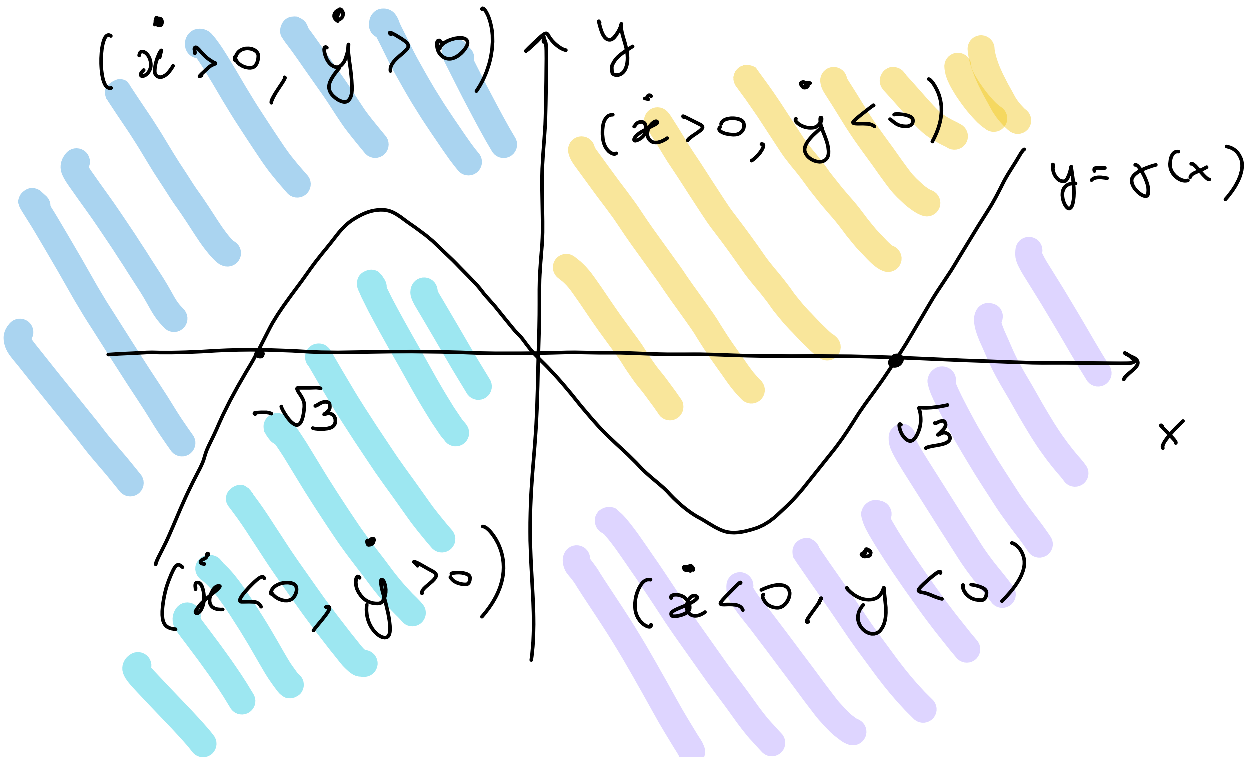
The equation at second order results then equivalent to this system:

$$\begin{cases} \dot{x} = \beta(y - \gamma(x)) \\ \dot{y} = -\dfrac{x}{\beta} \end{cases}$$

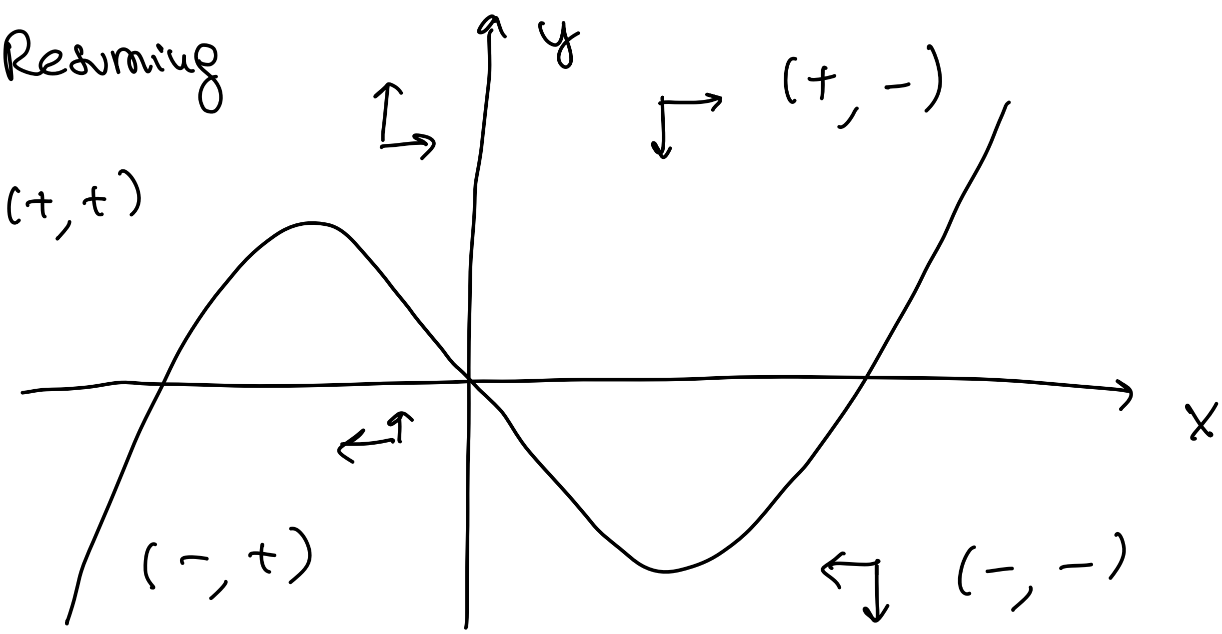
$\rightarrow$ where $\gamma(x) = \frac{x^3}{3} - x$ cubic

$\left[\frac{d}{dt} \beta y = -x \iff \dot{y} = -x/_{\beta} \right]$

Study the dynamics on the plane Oxy.



Resuming



Now we use the hypothesis that

$\beta >> 1$.

$$\alpha = \frac{\dot{y}}{\dot{x}} = -\frac{x}{\beta} \cdot \frac{1}{\beta(y-\gamma(x))} =$$

slope of the v.f.

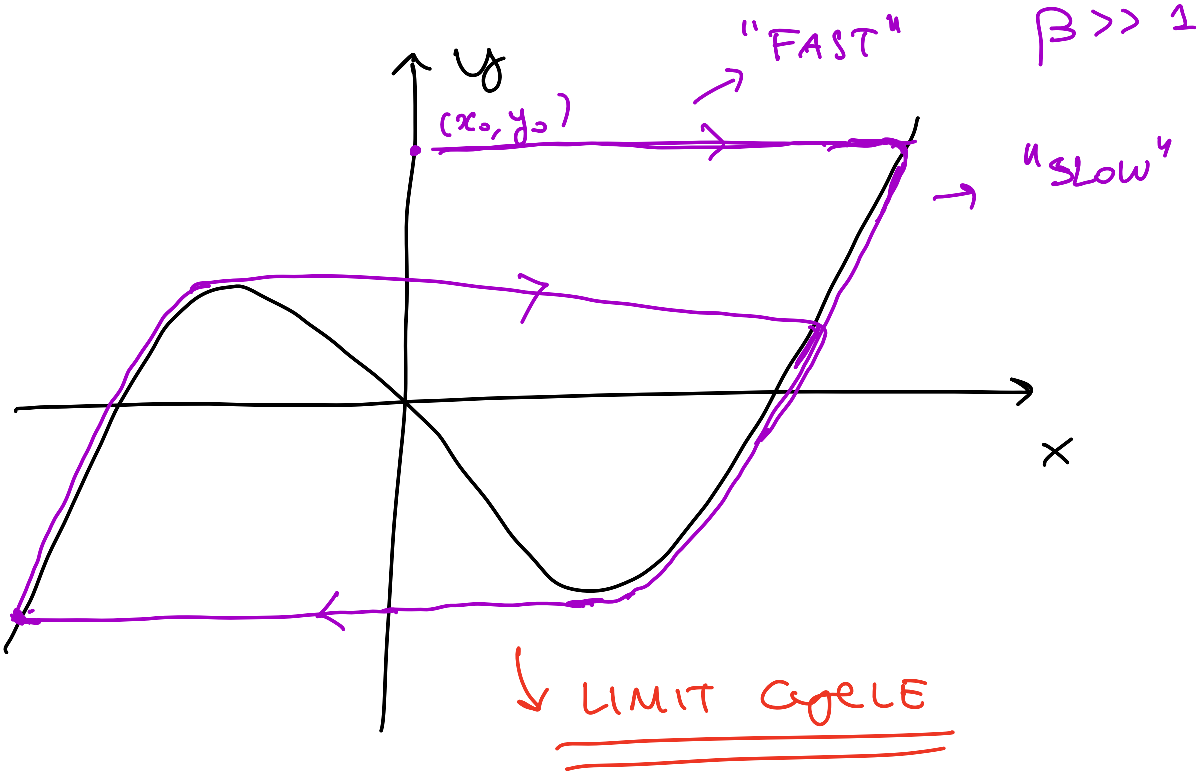
$$= -\beta^{-2} \frac{x}{y-\gamma(x)}$$

Consequently, FAR FROM $\gamma(x)$,
we have that $\alpha \simeq \frac{1}{\beta^2} << 1$

$\Longrightarrow$ FAR FROM $\gamma(x)$, trajectories
are "quite" horizontal.

"$\longrightarrow$" above $\gamma(x)$

"$\longleftarrow$" below $\gamma(x)$



$\beta \ll 1$ → we use a "perturbative" argument.

First order:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\beta(x^2-1)v - x \end{cases}$$

$\beta = 0$
$$\begin{cases} \dot{x} = v \\ \dot{v} = -x \end{cases}$$

Harmonic oscillator.

$$E = \frac{1}{2}(x^2 + v^2)$$

$$2E = x^2 + v^2 \longrightarrow \text{"}r\text{"} = \sqrt{2E}$$

$$\begin{cases} x = \sqrt{2E}\cos\vartheta \\ v = \sqrt{2E}\sin\vartheta \end{cases} \qquad \vartheta = \arctg \frac{v}{x}$$

$$\dot{E} = x\dot{x} + v\dot{v} =$$

$$= x(v) + v(-x - \beta(x^2-1)v) =$$

$$= \cancel{xv} - \cancel{xv} - \beta(x^2-1)v^2 =$$

$$= -\beta(2E\cos^2\vartheta - 1)\,2E\,\text{sen}^2\vartheta$$

$$= \beta\underbrace{(2E\sin^2\vartheta)(1 - 2E\cos^2\vartheta)}_{f(\vartheta,\,E)}$$

bounded if E bounded

$\hookrightarrow$ E is a slow variable.

$$\dot{E} = \beta f(\vartheta, E). \qquad \vartheta = \operatorname{arctg} \frac{v}{x}$$

$$\dot{\vartheta} = \frac{1}{1+\frac{v^2}{x^2}} \frac{(\dot{v}x - \dot{x}v)}{x^2} = \dots =$$

$$= -1 + \beta \overbrace{\left[-\sin\vartheta \cos\vartheta \left(2E\cos^2\vartheta - 1\right)\right]}^{g(E,\vartheta)}$$

$$\begin{cases} \dot{E} = \beta f(\vartheta, E) \longrightarrow \text{SLOW VARIABLE} \\ \dot{\vartheta} = -1 + \beta g(\vartheta, E) \longrightarrow \text{FAST VARIABLE} \end{cases}$$

Idea:

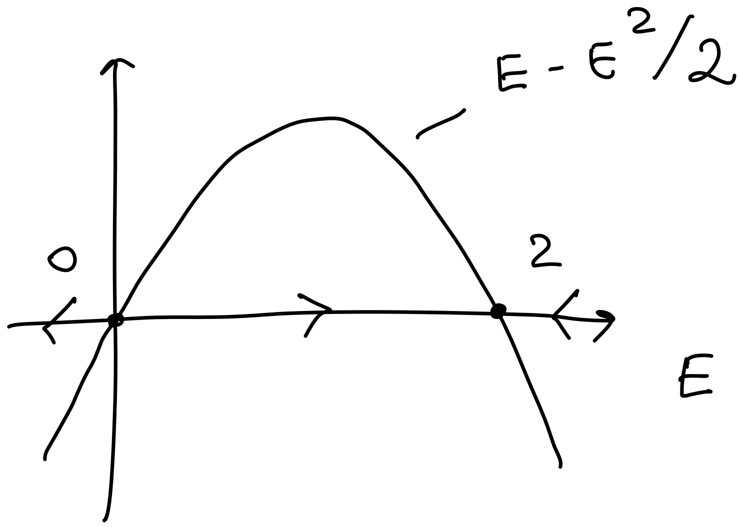
Substitute $f(\vartheta, E)$ with

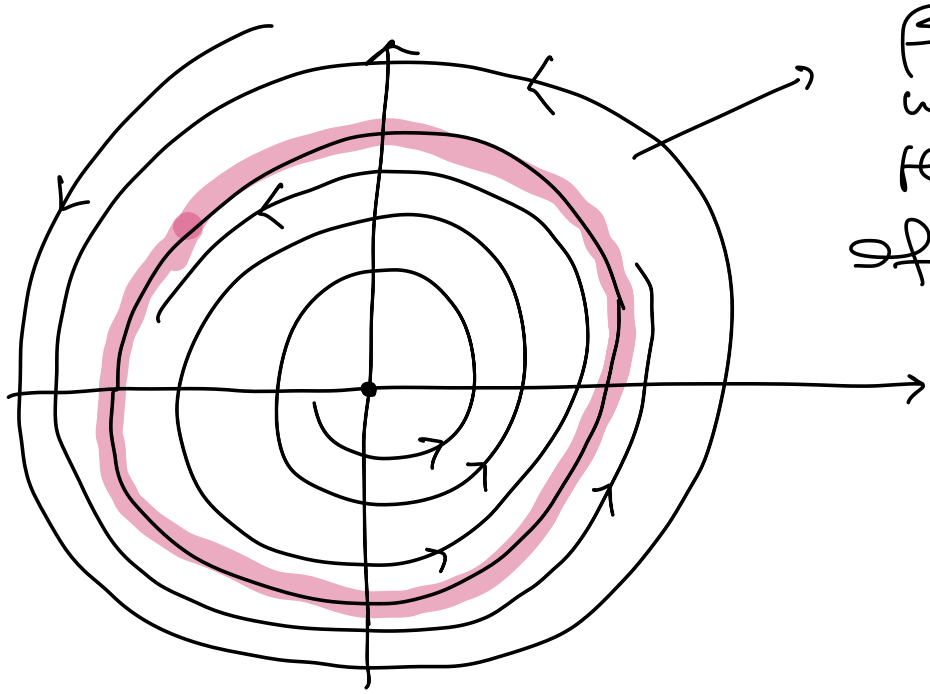
$$\bar{f}(E) = \frac{1}{2\pi}\int_0^{2\pi} f(E,\vartheta)\, d\vartheta$$

$$\bar{f}(E) = \dots = E - \frac{E^2}{2}$$

Approximated eq :

$$\begin{cases} \dot{E} = \beta\left(E - E^2/2\right) \\ \dot{\theta} = -1 + \beta g(E, \theta) \end{cases}$$

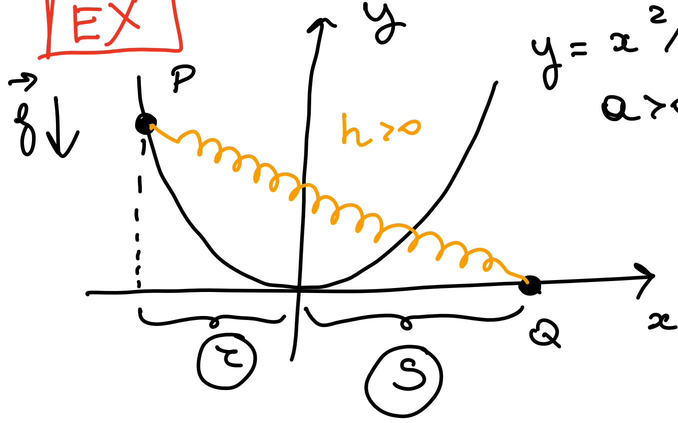




$\beta \ll 1$
we prove the existence of the limit cycle

EX - Solution -

EX



$y = x^2/a$

$a>0$

P, m

Q, m

$\tau = x_P$

$h = mg/a$

- $L(\tau, s, \dot{\tau}, \dot{s})$
- $h = mg/a$,
- $a = a(\tau, s)$?

Freq. of Small oscillations around the stable eq.

$$V_{el} = \frac{1}{2}k\left((-\tau+s)^2 + \frac{\tau^4}{a^2}\right)$$

$$\vec{OP} = (\tau, \tau^2/a) \to \left(\dot{\vec{OP}} = \left(\dot{\tau}, \frac{2\tau}{a}\dot{\tau}\right)\right)$$

$$\vec{OQ} = (s, 0)$$

$$V(\tau, s) = mg(\tau^2/a) + \frac{1}{2}k\left((-\tau+s)^2 + \frac{\tau^4}{a^2}\right)$$

$$= \frac{mg}{a}\tau^2 + \frac{1}{2}k\left(\tau^2 + s^2 - 2\tau s + \frac{\tau^4}{a^2}\right)$$

$$= \left(\frac{mg}{a} + \frac{1}{2}k\right)\tau^2 + \frac{k}{2}s^2 - k\tau s + \frac{k}{2a^2}\tau^4$$

Eq.

$$\nabla V(\tau, s) = \begin{pmatrix} \left(\frac{2mg}{a} + k\right)\tau - ks + \frac{2k}{a^2}\tau^3 \\ \\ \underline{ks - k\tau} \end{pmatrix}$$

$$\downarrow$$

$$\boxed{s = \tau}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\tau = 0$

$$\left(\frac{2mg}{a} + \frac{2k}{a^2}\tau^2\right)\tau = 0$$

$(0,0)$ $\exists !$ equilibrium.

$$\text{Hess } V(0,0) = \begin{pmatrix} \frac{2mg}{a} + k & -k \\ -k & k \end{pmatrix}$$

$\in \mathrm{Sym}^+ \Rightarrow (0,0)$ STABLE.

!!

$$K(\tau, s, \dot{\tau}, \dot{s}) =$$

$$= \frac{1}{2} m \left(\dot{\tau}^2 + \left(\frac{2\tau\dot{\tau}}{a} \right)^2 \right) + \frac{1}{2} m \dot{s}^2$$

$$= \frac{1}{2} m \left(1 + \frac{4\tau^2}{a^2} \right) \dot{\tau}^2 + \frac{1}{2} m \dot{s}^2$$

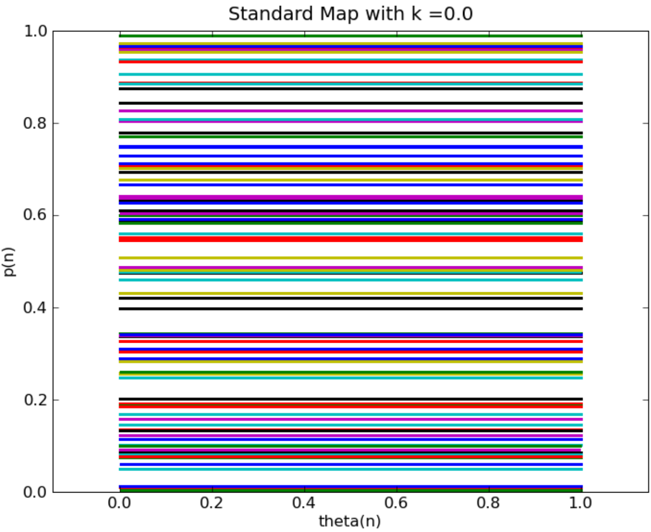
$$Q = \begin{pmatrix} m\left(1 + \frac{4\tau^2}{a^2}\right) & 0 \\ 0 & m \end{pmatrix}$$

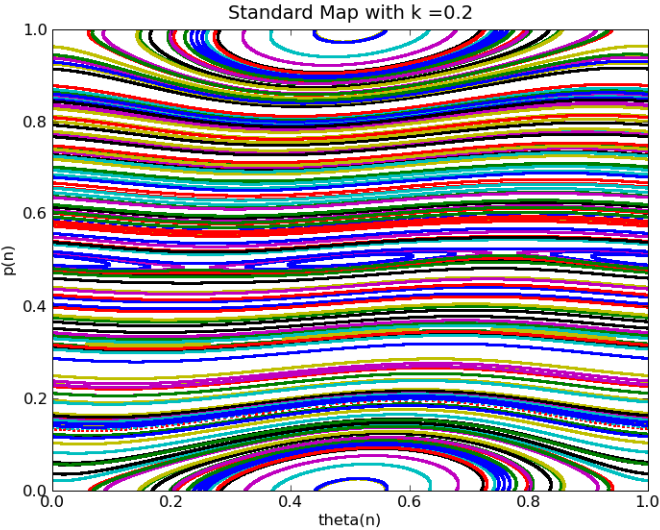
$$Q(0,0) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

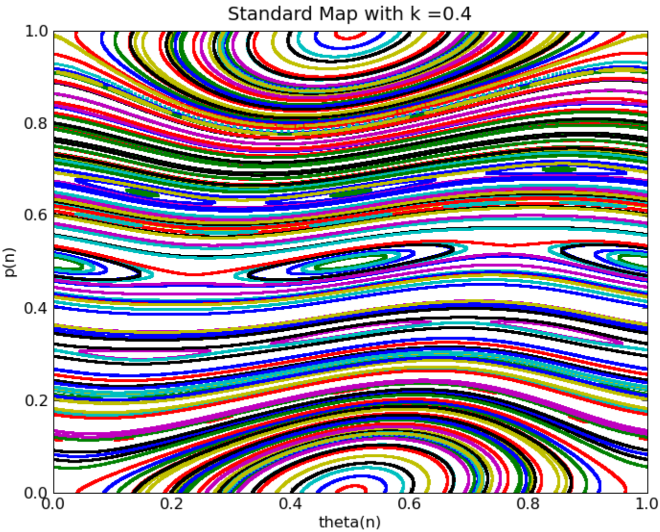
Freq. of small oscill. around $(0,0)$:

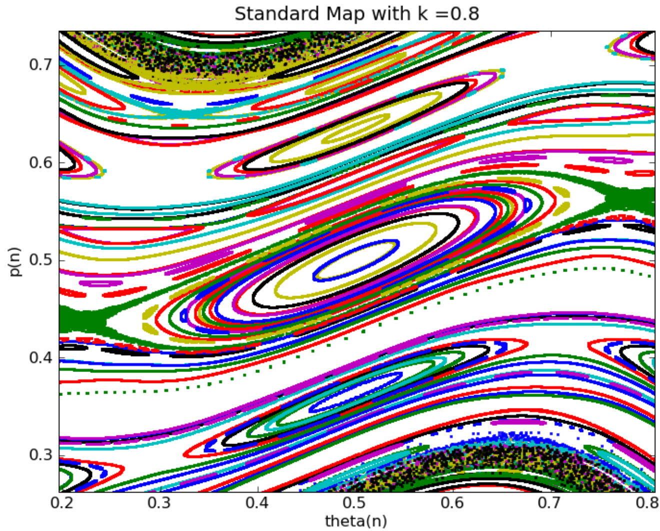
$$0 = \det \left(\text{Hess } V(0,0) - \omega^2 Q(0,0) \right)$$

$$\Leftrightarrow \boxed{\omega_{1,2} = \sqrt{(2 \pm \sqrt{2})\, k/m}}$$

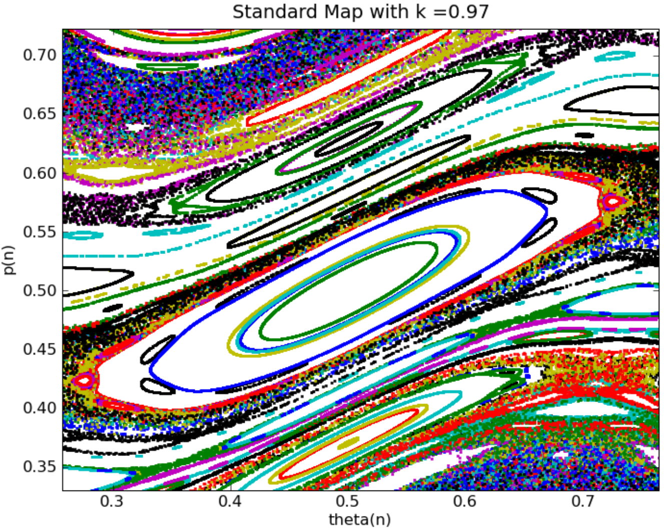

Standard Map with k =0.0
1.0
0.8
0.6
0.4
0.2
0.0
p(n)
0.0
0.2
0.4
0.6
0.8
1.0
theta(n)


Standard Map with k =0.2
1.0
0.8
0.6
0.4
0.2
0.0
p(n)
0.0
0.2
0.4
0.6
0.8
1.0
theta(n)


Standard Map with k =0.4
1.0
0.8
0.6
0.4
0.2
0.0
p(n)
0.0
0.2
0.4
0.6
0.8
1.0
theta(n)



Standard Map with k =0.8
0.7
0.6
0.5
0.4
0.3
p(n)
0.2
0.3
0.4
0.5
0.6
0.7
0.8
theta(n)



Standard Map with k =0.97
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
p(n)
0.3
0.4
0.5
0.6
0.7
theta(n)