Geometria 1 - mod. A - Lezione 22

Note Title

$$\mathcal{V}=\{v_1,\dots,v_n\}\quad V \xrightarrow{\ f\ } W \quad \mathcal{W}=\{w_1,\dots,w_m\}$$

$\alpha_{\mathcal{V}\mathcal{W}}(f) = A = (A_1, \dots, A_n)$

A_i = coordinate di $f(v_i)$ risp. alla base $\mathcal{W}$.

se $f = id_V$ (se $W=V$ e $f=id$)

$\alpha_{\mathcal{V}\mathcal{W}}(id)$ si dice matrice di cambio di base

$$\begin{tikzcd}
\mathcal{V} \quad V \arrow[r, "f", "A"'] & W \quad \mathcal{W} \arrow[d, "id_W"] \\
\mathcal{V}' \quad V \arrow[u, "id_V"] \arrow[r, "f"', "A'"] & W \quad \mathcal{W}'
\end{tikzcd}
\qquad P \qquad Q$$

$$\boxed{f = id_W \circ f \circ id_V} \ (*)$$
$$\Downarrow$$
$$A' = Q A P$$

$\mathcal{V}, \mathcal{V}'$ basi di V e $\mathcal{W}, \mathcal{W}'$ basi di W

$$\alpha_{\mathcal{V}'\mathcal{W}'}(f) \overset{?}{\longleftrightarrow} \alpha_{\mathcal{V}\mathcal{W}}(f)$$

$A' = \alpha_{\mathcal{V}'\mathcal{W}'}(f)$

$A = \alpha_{\mathcal{V}\mathcal{W}}(f)$

$P = \alpha_{\mathcal{V}'\mathcal{V}}(id_V)$ $\qquad$ $Q = \alpha_{\mathcal{W}\mathcal{W}'}(id_W)$

Da $(*)$ discende che $A' = QAP$

Esercizio: $\mathbb{R}^3 = U \oplus W$, $U = \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$, $W = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$

Scrivere la matrice rispetto alla base canonica di

$\pi_U^W, \ \pi_W^U, \ \sigma_U^W, \ \sigma_W^U, \ id_{\mathbb{R}^3}$

$$\alpha_{\mathcal{E}\mathcal{E}}(id_{\mathbb{R}^3}) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \mathbb{1}_3 \qquad \left(\text{sempre } \alpha_{\mathcal{V}\mathcal{V}}(id_V) = \mathbb{1}\right)$$

$v \in \mathbb{R}^3$

$$v = \begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\overset{\frac{x-z}{2}}{\cancel{a}}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \overset{y}{\cancel{b}}\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\in U} + \underbrace{\overset{\frac{x+z}{2}}{\cancel{c}}\begin{pmatrix}1\\0\\1\end{pmatrix}}_{\in W}$$

supposto noto

$$\begin{cases} a + c = x \\ b = y \\ -a + c = z \end{cases}$$

sistema nelle incognite a, b, c

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{x-z}{2} \\ y \\ -\frac{x-z}{2} \end{pmatrix}}_{= u} + \underbrace{\begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix}}_{= w}$$

$$\pi_U^W(v) = \begin{pmatrix} \frac{x-z}{2} \\ y \\ -\frac{x-z}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x & -\frac{1}{2}z \\ & y \\ -\frac{1}{2}x & +\frac{1}{2}z \end{pmatrix} \qquad \alpha_{\mathcal{E}\mathcal{E}}(\pi_U^W) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = A_U$$

$$\alpha_{\mathcal{E}\mathcal{E}}(\pi_W^U) = ? \quad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = A_W \qquad \pi_W^U(v) = \begin{pmatrix} \frac{x}{2} + \frac{z}{2} \\ 0 \\ \frac{x}{2} + \frac{z}{2} \end{pmatrix}$$

$$\boxed{\pi_U^W + \pi_W^U = id_{\mathbb{R}^3}}$$

$$A_U + A_W = \mathbb{1}_3$$

altro modo per calcolare A_W $\qquad A_W = \mathbb{1}_3 - A_U$

$$\alpha_{\mathcal{E}\mathcal{E}}(\sigma_U^W) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$u + w = v \longmapsto u - w$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \frac{x-z}{2} \\ y \\ -\frac{x-z}{2} \end{pmatrix} - \begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix} = \begin{pmatrix} -z \\ y \\ -x \end{pmatrix}$$

$$\sigma_U^W = \pi_U^W - \pi_W^U = A_U - A_W$$

$$\overset{!}{=} id - \pi_W^U - \pi_W^U = id - 2\pi_W^U = \mathbb{1}_3 - 2A_W$$

$$\alpha_{\mathcal{E}\mathcal{E}}(\sigma_W^U) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad v \longmapsto -u + w = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}$$

$$\mathbb{R}^3 = U \oplus W, \quad U = \left\langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{= v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{= v_2} \right\rangle, \quad W = \left\langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{= v_3} \right\rangle$$

$$= \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \langle v_3 \rangle$$

$$\mathcal{V} = \{v_1, v_2, v_3\}$$

$$\alpha_{\mathcal{V}\mathcal{V}}(\pi_U^W) = ? \quad \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} = C_U$$

$$v_1 = v_1 + 0_W \longmapsto v_1$$
$$v_2 \longmapsto v_2$$
$$v_3 = 0 + v_3 \longmapsto 0$$

$$v_1 = 1v_1 + 0v_2 + 0v_3, \quad v_2 = 0v_1 + 1v_2 + 0v_3$$

(È sempre vero che se $U \oplus W = V$ e considero una bas

v di V ottenuta unendo una base $v_1, \dots, v_r$ di U con una base $v_{r+1}, \dots, v_n$ di W allora

$$\alpha_{vv}(\pi_U^W) = \left(\begin{array}{c|c} \boxed{\begin{matrix} 1 & 0 & \cdots \\ 0 & \ddots & \\ & & 1 \end{matrix}} & 0 \\ 0 \cdots 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \boxed{\mathbb{1}_r} & 0 \\ \hline 0 & 0 \end{array}\right) = B$$

Torno all'esercizio

$$\alpha_{vv}(\pi_W^U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} = \mathbb{1}_3 - B \qquad \pi_W^U : \begin{array}{l} v_1 \longmapsto 0 \\ v_2 \longmapsto 0 \\ v_3 \longmapsto v_3 \end{array}$$

$$\pi_W^U = id - \pi_U^W$$

Nella ipotesi ⊛ $\alpha_{vv}(\pi_W^U) = \left(\begin{array}{c|ccc} 0 & 0 & & \\ \vdots & \vdots & & \\ & 1 & & \\ \cdot\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{array}\right)$ $\qquad \begin{array}{l} v_1 \longrightarrow 0 \\ \vdots \\ v_r \longrightarrow 0 \\ v_j \longrightarrow v_j \\ \text{se } j > r \end{array}$

$r \quad e_{r+1} \quad e_{r+2} \quad e_n$

$$= \left(\begin{array}{c|c} 0_{r\times r} & 0_{r \times n-r} \\ \hline 0_{n-r \times r} & \mathbb{1}_{n-r} \end{array}\right)$$

torno esercizio

$$\alpha_{vv}(\sigma_U^W) = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & 0 \\ & 0 \\ 0 \quad 0 & \boxed{-1} \end{pmatrix}$$

$$\begin{array}{l} u_1 \longmapsto u_1 = u_1 - 0 \\ \ \ \| \\ u_1 + 0 \\ u_2 \longmapsto u_2 \\ W \ni u_3 = 0 + u_3 \longmapsto -u_3 \end{array}$$

Nella ipotesi di ⊛ $\alpha_{vv}(\sigma_U^W) = \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & -\mathbb{1}_{n-r} \end{array}\right)$

$$\alpha_{vv}(\sigma_W^U) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad \text{in generale} \quad \left(\begin{array}{c|c} -\mathbb{1}_r & 0 \\ \hline 0 & \mathbb{1}_{n-r} \end{array}\right).$$

Torno

$$\begin{array}{ccc} \mathcal{E}\ \mathbb{R}^3 & \xrightarrow[A_U]{\pi_U^W} & \mathbb{R}^3\ \mathcal{E} \\ P \quad id \uparrow & & \downarrow id \quad Q = P^{-1} \\ \alpha_{v\mathcal{E}}(id) \quad \mathbb{R}^3 & \xrightarrow[C_U]{\pi_U^W} & \mathbb{R}^3 \quad \alpha_{\mathcal{E}v}(id) \\ v & & v \end{array}$$

Dalla teoria $C_U = Q A_U P$

$$C_U = P^{-1} A_U P \qquad P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Troviamo inversa con l'algor. di Gauss.

$v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1e_1 + 0e_2 - 1e_3$