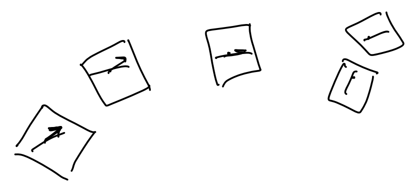
Lesson 23 - 21/11/2022

- Complete proof of L-D Theorem
- Konig Theorem
- Inertia matrices for : BAR, RING, DISC, RECTANGLE (SQUARE)
- 1 EX. on Lagrangian formulation.

$P_1 \dots P_N$ subj. to $\vec{F}_1 \dots \vec{F}_N$ forces.

$q_1 - q_m$ Lagrange coordinates,

$$Q_h = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} \qquad \forall\ h = 1 \_ m.$$



As a consequence, the work of these system of forces corresponding to virtual displacements of points $P_1 \_ P_N$:

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{OP}_i = \sum_{i=1}^{N} \vec{F}_i \cdot \sum_{h=1}^{m} \frac{\partial \vec{OP}_i}{\partial q_h} \delta q_h =$$

$$= \sum_{i=1}^{m} \underbrace{\sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h}}_{Q_h} \delta q_h = \sum_{h=1}^{m} Q_h \delta q_h .$$

Proof of L-D Theorem

Use $E(q,\dot{q}) = K(q,\dot{q}) + V(q) - V(q^*)$ as a Lyapunov function in order to prove stability of $(q^*, 0)$.

- Since q^* is a strict minimum for V, then E is positive definite in a neighborhood of $(q^*, 0)$.

$(q^*, 0)$

$K.$ $\dot{q} = 0$ $\Downarrow$ $K(q, 0) \equiv 0.$

- Lie derivative of E.

$$\frac{d}{dt}\left(\frac{1}{2} m |v|^2\right) + \frac{d}{dt} V =$$

$$= m \vec{v} \cdot \vec{a} + \dot{V} =$$

$$= (\vec{F} + \vec{\Phi}) \cdot \vec{v} + \dot{V} =$$

$\longrightarrow$ ideal constraints ($\Rightarrow \equiv 0$)

$$= \sum_{i=1}^{m} \left(Q_n^1(q) + Q_n^2(q,\dot q) \right) \dot q_n - \sum_{i=1}^{m} Q_n^1(q)\, \dot q_n$$

$$= \sum_{i=1}^{m} Q_n^2(q,\dot q)\, \dot q_n \leq 0 \quad \text{by hypotesis.}$$

$\Rightarrow$ $(q^*, 0)$ is stable (topologically) $\square$

—x—x—

Rigid systems

S = rigid system of points. Then

$$\vec{v}_i = \vec{v}_J + \vec{\omega} \wedge \overrightarrow{P_J P_i}$$ $\longrightarrow$ fundamental formula for rigid motions.

In particular, we can choose for $\vec{v}_J$ the velocity of the center of mass, defined as follows:

$$\vec{OG} = \frac{\sum_{i=1}^{N} m_i \vec{OP_i}}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{OP_i}}{m}$$

$$\vec{V_i} = \vec{V_G} + \vec{\omega} \wedge \vec{GP_i}$$

Then

$$\sum_{i=1}^{N} m_i \vec{GP_i} = \sum_{i=1}^{N} m_i \left(\vec{OP_i} - \vec{OG} \right) = m \vec{OG} - m \vec{OG} \Rightarrow$$

$$\sum_{i=1}^{N} m_i \vec{GP_i} \equiv 0.$$

- Kinetic energy of the rigid system $P_1 \text{—} P_N$

$$2K = \sum_{i=1}^{N} m_i |\vec{v}_i|^2 = \sum_{i=1}^{N} m_i \left| \vec{v}_G + \vec{\omega} \wedge \vec{GP_i} \right|^2 =$$

$\downarrow$

fundamental formula of rigid motions.

$$= \sum_{i=1}^{N} m_i |\vec{v}_G|^2 + 2 \sum_{i=1}^{N} m_i \vec{v}_G \cdot (\vec{\omega} \wedge \vec{GP}_i) +$$

$$+ \sum_{i=1}^{N} m_i |\vec{\omega} \wedge \vec{GP}_i|^2 =$$

$$= m |\vec{v}_G|^2 + \underbrace{2 \vec{V}_G \cdot \vec{\omega} \wedge \underbrace{\sum_{i=1}^{N} m_i \vec{GP}_i}}_{\equiv 0} + \sum_{i=1}^{N} m_i |\vec{\omega} \wedge \vec{GP}_i|^2 =$$

$$= m |\vec{v}_G|^2 + \sum_{i=1}^{N} m_i (\vec{\omega} \wedge \vec{GP}_i) \cdot (\vec{\omega} \wedge \vec{GP}_i)$$

$$\left[\vec{a} \cdot (\vec{b} \wedge \vec{c}) = \vec{b} \cdot (\vec{c} \wedge \vec{a}). \text{ In our case: } (\vec{\omega} \wedge \vec{GP}_i) \cdot (\vec{\omega} \wedge \vec{GP}_i) = \vec{\omega} \cdot [\vec{GP}_i \wedge (\vec{\omega} \wedge \vec{GP}_i)] \right]$$

$$= m |\vec{v}_G|^2 + \sum_{i=1}^{N} m_i \vec{\omega} \cdot [\vec{GP}_i \wedge (\vec{\omega} \wedge \vec{GP}_i)]$$

$$\left[\vec{a} \wedge (\vec{b} \wedge \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \right]$$

$$= m |\vec{v}_G|^2 + \vec{\omega} \sum_{i=1}^{N} m_i \left[|\vec{GP}_i|^2 \vec{\omega} - \underbrace{(\vec{GP}_i \cdot \vec{\omega})}_{\in \mathbb{R}} \vec{GP}_i \right] =$$

$$= m |\vec{v}_G|^2 + \vec{\omega} I_G \vec{\omega}$$

$$(I_G)_{hk} = \sum_{i=1}^{N} m_i \left[|\vec{GP}_i|^2 \delta_{hk} - x_h^i x_k^i \right]$$

with $\vec{GP}_i = (x_1^i, x_2^i, x_3^i)$.

Konig Theorem

Rigid system. G center of mass.

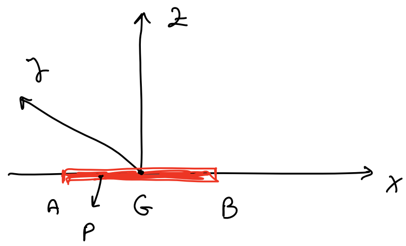
$$\boxed{K = \frac{1}{2} \left[m |\vec{v}_G|^2 + \vec{\omega} I_G \vec{\omega} \right].}$$

Inertial matrices for BAR, RING, DISC, RECTANGLE (SQUARE).

For rigid bodies: $m = \int_V \rho \, dV$

$$\boxed{(I_G)_{hk} = \int_V \rho \left[|\vec{GP}|^2 \delta_{hk} - x_h^P x_k^P \right] dV}$$

BAR



$\vec{GP} = (x, 0, 0)$

$x \in [-e/2, e/2]$

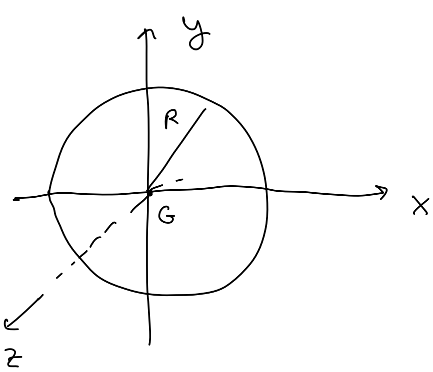
$$I_{11} = 0$$

$$I_{22} = \int_{-e/2}^{e/2} \rho x^2 \, dx = \int_{-e/2}^{e/2} \frac{m}{e} x^2 \, dx = \dots = \frac{me^2}{12}$$

$I_{22} = I_{33}$

$$I_G = \begin{pmatrix} 0 & & \\ & \frac{me^2}{12} & \\ & & \frac{me^2}{12} \end{pmatrix}$$

RING



$\vec{GP} = (R\cos\theta, R\sin\theta, 0)$

$\theta \in [0, 2\pi[$.

$$I_{11} = \int_0^{2\pi} \rho \left(R^2 - R^2 \cos^2\vartheta \right) R \, d\vartheta =$$

$$= \int_0^{2\pi} \frac{m}{2\pi R} R^2 \underbrace{\left(1 - \cos^2\vartheta \right)}_{= \sin^2\vartheta} R \, d\vartheta =$$

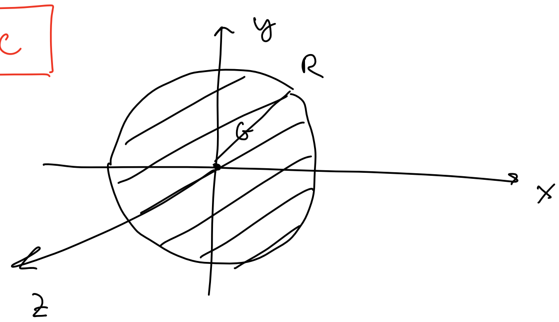
$$= \frac{m R^2}{2\pi \cancel{R}} \cdot \cancel{R} \underbrace{\int_0^{2\pi} \sin^2\vartheta \, d\vartheta}_{\pi} = \frac{m R^2 \cancel{\pi}}{2 \cancel{\pi}} = \frac{m R^2}{2}$$

$$I_{22} = I_{11}$$

$$I_{33} = \int_0^{2\pi} \rho \left(R^2 - 0 \right) R \, d\vartheta = \frac{m}{\cancel{2\pi} R} \cdot R^3 \cdot \cancel{2\pi} = m R^2$$

$$I_G = \begin{pmatrix} \frac{mR^2}{2} & & \\ & \frac{mR^2}{2} & \\ & & mR^2 \end{pmatrix}$$

$\boxed{\text{Disc}}$



$$\vec{GP} = \left(r\cos\vartheta, r\sin\vartheta, 0 \right)$$
$$r \in [0, R]$$
$$\vartheta \in [0, 2\pi)$$

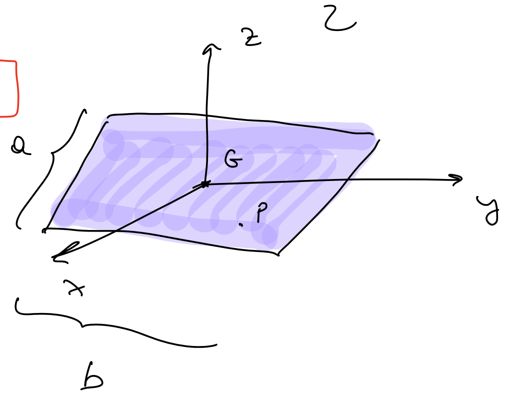
$$I_{11} = \int_0^{2\pi} \int_0^{R} \rho \left[r^2 - r^2 \cos^2\vartheta \right] r \, dr \, d\vartheta$$

$$= \int_0^{2\pi} \int_0^{R} \frac{m}{\pi R^2} \left[r^3 \sin^2\vartheta \right] dr \, d\vartheta =$$

$$= \frac{m}{\cancel{\pi} \cancel{R^2}} \cancel{\pi} \cdot \frac{1}{4} R^{\cancel{4}\,2} = \frac{m R^2}{4}$$

$$I_G = \begin{pmatrix} \frac{mR^2}{4} & & \\ & \frac{mR^2}{4} & \\ & & \frac{mR^2}{2} \end{pmatrix}$$

RECTANGLE



$$-\frac{a}{2} \leq x \leq \frac{a}{2}$$

$$-\frac{b}{2} \leq y \leq \frac{b}{2}$$

$$\vec{GP} = (x, y, 0)$$

$$I_{11} = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \frac{m}{ab} \left[\cancel{x^2} + y^2 - \cancel{x^2} \right] dx\,dy$$

$$= \frac{m}{\cancel{a}b} \frac{1}{3} y^3 \Big|_{-b/2}^{+b/2} \cancel{a} = \frac{mb^2}{12}$$

$$I_{22} = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \frac{m}{ab} \left[x^2 + y^2 - y^2 \right] dx\,dy$$

$$= \frac{ma^2}{12}$$

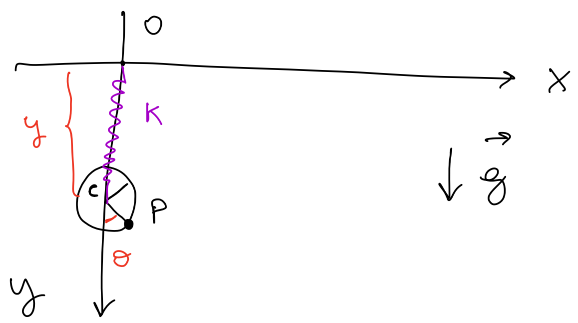
$$I_{33} = I_{11} + I_{22} = \frac{m(a^2+b^2)}{12}$$

$$I_G = \begin{pmatrix} mb^2/12 & & \\ & ma^2/12 & \\ & & m(a^2+b^2)/12 \end{pmatrix}$$

For the square : $a = b$

$$I_G = \begin{pmatrix} ma^2/12 & & \\ & ma^2/12 & \\ & & ma^2/6 \end{pmatrix}$$

$\boxed{\text{EX 1}}$



Disc M, R

The disc doesn't rotate!!

P, m moves along the boundary of the disc.

- Lagrangian
- Equilibria & stability.

$$\vec{OC} = (0, y). \qquad \vec{v}_c = \dot{y}^2$$

$$\vec{OP} = (R\sin\theta, \underbrace{y + R\cos\theta}_{\dot{y} - R\dot{\theta}\sin\theta})$$

$$\vec{v}_P^{\,2} = R^2\dot{\theta}^2 + \dot{y}^2 - 2R\dot{y}\dot{\theta}\sin\theta$$

$$K = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} m \left(\dot{y}^2 + R^2\dot{\theta}^2 - 2R\sin\theta\, \dot{y}\dot{\theta} \right)$$

$$V = \frac{1}{2} k |\vec{OC}|^2 - M g y_c - mg\, y_P$$

$$= \frac{1}{2} k y^2 - M g y - mg (y + R\cos\theta)$$

$L = K - V = L(y, \theta, \dot{y}, \dot{\theta})$ No cyclic coo.

$\Rightarrow$ The unique conserved quantity is $\underline{E = K + V}$

$$\nabla V = (\partial_y V, \partial_\theta V) = (ky - Mg - mg, \; mgR\sin\theta)$$

$$\begin{cases} y = \dfrac{g(M+m)}{k} \\ \theta = 0, \pi \end{cases}$$

EQ. conf. $\underbrace{\left(\dfrac{g(M+m)}{k}, 0\right)}_{EQ_1}$ and $\underbrace{\left(\dfrac{g(M+m)}{k}, \pi\right)}_{EQ_2}$

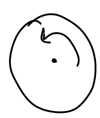
$$\text{Hess } V = \begin{pmatrix} k & 0 \\ 0 & mgR\cos\theta \end{pmatrix}$$

$$\text{Hess } V(EQ_1) = \begin{pmatrix} k & 0 \\ 0 & mgR \end{pmatrix} \to \text{STABLE}$$

$$\text{Hess } V(EQ_2) = \begin{pmatrix} k & 0 \\ 0 & -mgR \end{pmatrix} \to \text{UNSTABLE}$$

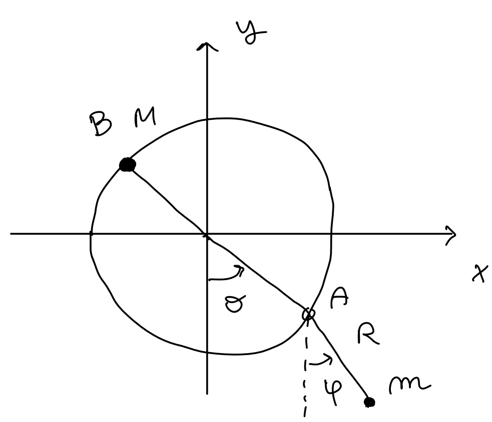
EX 2

Ring M, R can rotate around its central orthogonal axis



Pendulum $\underline{m < M}$ lenght R in a point A of the ring.
Point M fixed in the point B of the ring,
B opposite to A.

$\downarrow \vec{g}$



Lagr. coord. θ, φ.

- Eq. stability
- Kinetic matrix.

Potential energy of the system.

$$V = \underline{MgR\cos\vartheta} - mgR(\underline{\cos\vartheta} + \cos\varphi) =$$

$$= (M-m)gR\cos\vartheta - mgR\cos\varphi$$

$$\frac{\partial V}{\partial\vartheta} = -(M-m)gR\sin\vartheta = 0$$

$$\frac{\partial V}{\partial\varphi} = mgR\sin\varphi = 0$$

$$(\vartheta^*, \varphi^*) = (0,0), (0,\pi), (\pi,0), (\pi,\pi).$$

$$\mathrm{Hess}\,V(\vartheta,\varphi) = \begin{pmatrix} \underbrace{-(M-m)}_{>0}\cos\vartheta & 0 \\ 0 & m\cos\varphi \end{pmatrix} gR$$

Stability when all entries are > 0 $(M > m)$

$\Rightarrow$ The unique stable eq. is $\underline{\underline{(\pi, 0)}}$.

<u>Kinetic energy</u>

$$K_{RING} = \frac{1}{2} MR^2 \dot\vartheta^2 \qquad \left(\frac{1}{2} \vec\omega\, \underbrace{I_G}_{\begin{pmatrix} mR^2/2 & & \\ & mR^2/2 & \\ & & mR^2 \end{pmatrix}} \vec\omega \qquad \vec\omega = \begin{pmatrix} 0 \\ 0 \\ \dot\vartheta \end{pmatrix} \right)$$

$$K_B = \frac{1}{2} MR^2\dot\vartheta^2$$

$$K_m = \_ = \frac{1}{2} mR^2 \left[\dot\vartheta^2 + \dot\varphi^2 + 2\cos(\vartheta-\varphi)\dot\vartheta\dot\varphi \right]$$

$\vec{OP} = \ldots$ and then derivative wrt time ...

$a(\vartheta, \varphi)$.