Remind:

**Weirstrass Theorem:** $f: [a,b] \rightarrow \mathbb{R}$ continuous, then there exist a (global) maximum and a (global) minimum.

**Fermat Theorem:** If $f: D \rightarrow \mathbb{R}$, if $\xi \in \text{Int}(D)$ if differentiable at $\xi$, $\xi$ a relative maximum or minimum, then $f'(\xi) = 0$.
Other fundamental theorems of differential calculus:

Theorem (Rolle):
Let \( f: [a, b] \to \mathbb{R} \) be continuous on \([a, b] \)
- differentiable on \([a, b] \)
- such that \( f(a) = f(b) \)

Then there exists \( \exists \xi \in ]a, b[ \) such that \( f'(\xi) = 0 \)
Proof: By W. we know that there exists a minimum and a maximum.

\[ \min f = f(x_m) \]
\[ \max f = f(x_n) \]
\[ f(x_m) = f(x_n) \]

\[ \Rightarrow f \text{ is constant} \]
\[ \Rightarrow \exists \xi \in ]a,b[ , f'(\xi) = 0 \]

If case \[ f(x_m) < f(x_n) \]

\( x_m \) and \( x_n \) cannot be both end points because \( f(a) = f(b) \). \( x_m \) or \( x_n \) is \( \exists \xi \in ]a,b[ \) \[ \Rightarrow f'(\xi) = 0 \]

by Fermat q.e.d.
Theorem (Lagrange). Let \( f: [a, b] \to \mathbb{R} \) be
- continuous on \([a, b]\)
- differentiable on \([a, b]\).
Then there exists \( \xi \in ]a, b[ \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(\xi)
\]
Proof: \( g: [a,b] \rightarrow \mathbb{R} \)

\[ g(x) = f(x) - \frac{f(b) - f(a)}{b-a} \]

\[ g(a) = f(a) - \frac{f(b) - f(a)}{b-a} \]

\[ = \frac{f(a) - f(b) + f(a)}{b-a} \]

\[ = \frac{(f(a) - f(b))}{b-a} \]

\[ g(b) = f(b) - \frac{f(b) - f(a)}{b-a} \]

\[ = \frac{f(b) - f(b) + f(a)}{b-a} \]

\[ = \frac{(f(a) - f(b))}{b-a} \]

\[ \text{i.e. } g(a) = g(b) \]

\( g \) is continuous (sum of continuous)

\( g \) is differentiable on \([a,b] \).
Verify Rolle's hypothesis.

Rolle's Theorem:

\[ \exists c \in [a, b] \text{ s.t.} \]

\[ g'(c) = 0 \]

\[ g'(c) = f'(c) = \frac{f(b) - f(a)}{b - a} \]

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

q.e.d.
Some consequences of Lagrange Theorem:

**Theorem 1.** (Monotonicity and derivative's sign).

Let $I$ be any interval, and let $f: I \rightarrow \mathbb{R}$ a continuous function differentiable on $\text{int}(I)$.

1. $f$ is increasing $\iff f'(x) > 0 \ \forall x \in \text{int}(I)$
2. $f$ is decreasing $\iff f'(x) < 0 \ \forall x \in \text{int}(I)$
3. $f'(x) > 0 \ \forall x \in \text{int}(I) \implies f$ is strictly increasing
4. $f'(x) < 0 \ \forall x \in \text{int}(I) \implies f$ is strictly decreasing

**Examples**

- $f(x) = x^3, \ x \in [0, \infty)$
- $f(x) = x^{a-1}, \ a \in \mathbb{R}$
- $f(x) = \alpha x^{a-1} \ \forall x \neq 0$
\[ f(x) = \log x \quad f'(x) = \frac{1}{x} > 0 \Rightarrow \log x \text{ is strictly increasing} \]

\[ f(x) = e^x \quad f'(x) = e^x > 0 \Rightarrow e^x \text{ is strictly increasing} \]

\[ f(x) = x^3 - 4x^2 + 2x + 7 \]

\[ f'(x) = 3x^2 - 8x + 2 \]

\[ 3x^2 - 8x + 2 \geq 0 \quad x \in \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 24}}{6} \]

\[ x_1, \quad x_2, \quad x_3 \]

\[ \text{increasing} \rightarrow \text{decreasing} \rightarrow \text{increasing} \]

\[ f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2} \]

\[ f'(x) = e^x - \left( -e^{-x} \right) = e^x + e^{-x} = \cosh(x) > 0 \]
Exercise

Study the domain, symmetry, sign, limits, asymptotes, monotonicity of

\[ f(x) = \tan(x^2) \]

d and draw a qualitative graph

**Domain:** \( \{ x \in \mathbb{R} : x^2 \neq \frac{\pi}{2} + \pi k \} \) \( \quad k \in \mathbb{N} \)

\[ \{ x \in \mathbb{R} : x \neq \sqrt{\frac{\pi}{2} + \pi k} \} = \mathbb{D} \]

Even function: \( f(x) = f(-x) \quad \forall x \in \mathbb{D} \)

Odd function: \( f(x) = -f(-x) \)

\( f(-x) = \tan((-x)^2) = \tan(x^2) = f(x) \)
is even. Let us study if only on \( \mathbb{D} \cup [0, +\infty) \)

\[
\lim_{x \to \sqrt{\pi}^-} f(x) = \lim_{x \to \sqrt{\pi}^-} \csc^2(x^2) = \lim_{y \to \sqrt{3}} \csc^2(y) = +\infty
\]

\[
\lim_{x \to \sqrt{\pi}^+} f(x) = -\infty
\]

\[
\lim_{x \to \sqrt{\pi} \mp \sqrt{3} \pm} f(x) = -\infty
\]

\[
f'(x) = \frac{1}{\cos^2(x^2)}
\]

for \( x \in \mathbb{D} \cup [0, +\infty) \) \( f'(x) > 0 \)

\( \Rightarrow f \) is increasing on \( \mathbb{D} \cup [0, +\infty) \)

\[
\lim_{x \to +\infty} f(x) = 0
\]

\[
\lim_{k \to +\infty} f(\sqrt{k^2 + \pi}) = \lim_{k \to +\infty} \csc(\pi k) = 0
\]
\[
\lim_{x \to \infty} f \left( \sqrt{\sqrt{x} + x} \right) = \lim_{k \to 0^+} f \left( k \sqrt{\sqrt{x} + x} \right) = 1 \to 1
\]

\[\Rightarrow \lim_{x \to \infty} f(x) \text{ doesn't exist.}\]
Theorem 1. (Monotonicity and derivatives sign).

Let $I$ be any interval, and let $f: I \to \mathbb{R}$ a continuous function differentiable on $\text{int}(I)$.

1) $f$ is increasing $\iff f'(x) \geq 0 \forall x \in \text{int}(I)$
2) $f$ is decreasing $\iff f'(x) < 0 \forall x \in \text{int}(I)$
3) $f'(x) > 0 \forall x \in \text{int}(I) \implies f$ is strictly increasing.
4) $f'(x) < 0 \forall x \in \text{int}(I) \implies f$ is strictly decreasing.

Proof. \quad 1) \quad \Rightarrow \quad \frac{f(x+h) - f(x)}{h} \geq 0 \text{ if } h > 0 \quad \Rightarrow \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \geq 0 \quad \Rightarrow \quad f'(x) \geq 0$