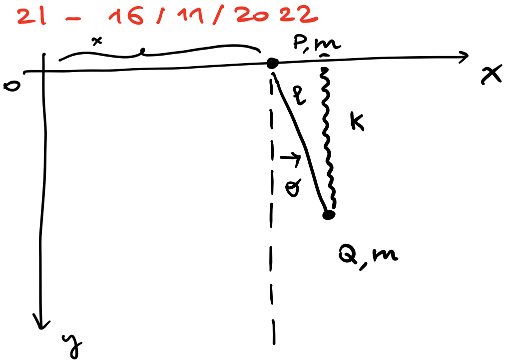
Lesson 21 - 16/11/2022

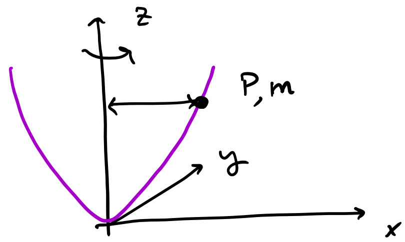
Ex 1



$\downarrow \vec{g}$

- Lagrangian
- Equilibria & stability
- First integral(s).

Ex 2



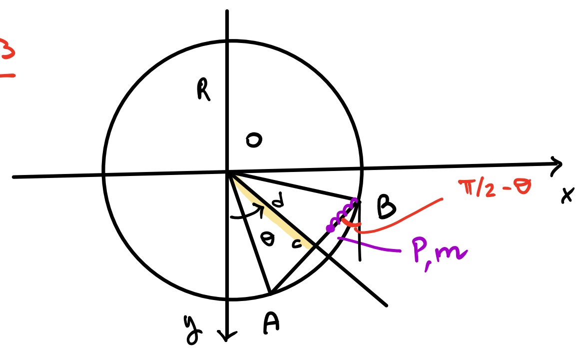
$y = x^2/2$ (parabola)

Lagrangian coordinate : x.

$\omega = \omega \hat{z}$

Eqs of motions by using both the fixed and rotating system.

Ex 3



R radius

K spring constant

$|AB| = 2\ell$, negl. mass

Lagrangian (V_{el})

First integral(s).

$$d = \sqrt{R^2 - \ell^2} \Rightarrow \vec{OC} = (d \sin\theta, d\cos\theta)$$

$$\vec{OB} = (d\sin\theta + \ell \sin(\pi/2 - \theta), d\cos\theta - \ell\cos(\pi/2 - \theta))$$

$$= (d\sin\theta + \ell\cos\theta, d\cos\theta - \ell\sin\theta)$$

Finally

$$\vec{OP} = (d\sin\theta + \ell\cos\theta - s\cos\theta, d\cos\theta - \ell\sin\theta + s\sin\theta)$$

Therefore

$$\vec{v}_P = (d\dot\theta\cos\theta - \ell\dot\theta\sin\theta + s\dot\theta\sin\theta - \dot s\cos\theta, -d\dot\theta\sin\theta - \ell\dot\theta\cos\theta + s\dot\theta\cos\theta + \dot s\sin\theta)$$

Hence

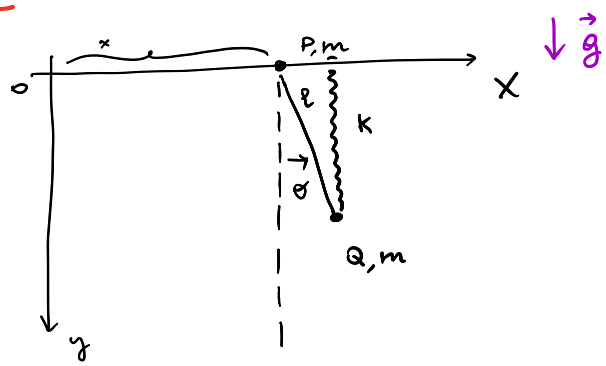
$$|\vec{v}_P|^2 = \dots = d^2\dot\theta^2 + \ell^2\dot\theta^2 + \dot s^2 + s^2\dot\theta^2 - 2\ell s\dot\theta^2 - 2d\dot\theta\dot s$$

↓ For tomorrow!

- Normal form of Lagrange eqs
- Cyclic coordinates
- Equilibria and their stability for mechanical Lagrangians.

SOLUTIONS

EX 1



x, θ Lagr. coord.

$\vec{OP} = (x, 0) \Rightarrow \vec{v}_P = (\dot{x}, 0)$

$\vec{OQ} = (x + \ell \sin\theta, \ \ell \cos\theta)$

$\vec{v}_Q = (\dot{x} + \ell \dot{\theta} \cos\theta, \ -\ell \dot{\theta} \sin\theta)$

$K = K_P + K_Q = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + \ell^2 \dot{\theta}^2 + 2 \ell \cos\theta \, \dot{\theta} \, \dot{x})$

$$= m\dot{x}^2 + \frac{1}{2} m e^2 \dot{\theta}^2 + m e \cos\theta\, \dot{\theta}\, \dot{x} =$$

$$= \frac{1}{2} \begin{pmatrix} 2m & ml\cos\theta \\ ml\cos\theta & me^2 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}$$

Potential energy

$$V = V_{gr} + V_{el} = -mge\cos\theta + \frac{1}{2} k (l\cos\theta)^2$$

Therefore, the Lagr is $L = K - V =$

$$= m\dot{x}^2 + \frac{1}{2} m e^2 \dot{\theta}^2 + m e \cos\theta\, \dot{\theta}\, \dot{x} +$$

$$+ mge\cos\theta - \frac{1}{2} k (e\cos\theta)^2 = L(\theta, \dot{\theta}, \dot{x})$$

not depend on x explicitly.

Lagr. eqs. are

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \end{cases}$$

$$2m\ddot{x} - me\sin\theta(\dot{\theta})^2 + me\cos\theta\,\ddot{\theta}$$

$$\begin{cases} \frac{d}{dt}\left(2m\dot{x} + me\cos\theta\,\dot{\theta}\right) = 0 \\ \frac{d}{dt}\left(me^2\dot{\theta} + me\cos\theta\,\dot{x}\right) + mge\sin\theta - ke\cos\theta\sin\theta = 0 \end{cases}$$

$$\hookrightarrow me^2\ddot{\theta} - me\sin\theta\,\dot{\theta}\dot{x} + me\cos\theta\,\ddot{x}$$

Equilibria & stability

$$\begin{cases} V_x \equiv 0 \\ V_\vartheta = mg\ell \sin\vartheta - k\ell^2 \cos\vartheta \sin\vartheta = 0 \end{cases}$$

$$\downarrow \underbrace{\ell \sin\vartheta}_{\downarrow} (mg - k\ell \cos\vartheta) = 0$$

$\vartheta = 0$ OR $\vartheta = \pi$

AND $mg = k\ell \cos\vartheta \rightarrow \vartheta_3 = \arccos\left(\frac{mg}{k\ell}\right)$

$\vartheta_4 = -\vartheta_3$

WHEN $-1 < \frac{mg}{k\ell} < 1$

EQUILIBRIA Configurations | $(x, 0)$, (x, π), (x, ϑ_3), $(x, \vartheta_4 = -\vartheta_3)$

$\forall x \in \mathbb{R}$.

Stability? We go to check the 2nd derivatives.

$$V_{\vartheta\vartheta} = mg\ell \cos\vartheta + k\ell^2 \sin^2\vartheta - k\ell^2 \cos^2\vartheta$$

$$V_{\vartheta x} \equiv 0$$

$$V_{xx} \equiv 0$$

$V_{\vartheta\vartheta}(0) = mg\ell - k\ell^2 < 0$ UNSTABLE EQ.

$V_{\vartheta\vartheta}(\pi) = -mg\ell - k\ell^2 < 0$ ALWAYS UNSTABLE EQ.

$$V_{\theta\theta}(\theta_3) = V_{\theta\theta}(\theta_4) = \dots = \frac{k^2e^2 - m^2g^2}{K} < 0$$

$K > 0$

iff $k^2e^2 - m^2g^2 < 0$

iff $k^2e^2 < m^2g^2$

iff $ke < mg$ iff $\frac{mg}{ke} > 1$ (or $\frac{ke}{mg} < 1$)

(From here we cannot conclude that it's unstable).

FIRST INTEGRALS

$\boxed{E = K + V}$

From $L(\theta, \dot\theta, \dot x) \Rightarrow L$ does not dep. on x

(x is called cyclic coordinate).

$$\frac{d}{dt}\frac{\partial L}{\partial \dot x} - \underbrace{\frac{\partial L}{\partial x}}_{\substack{\parallel \\ 0}} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) = 0$$

0 since x cyclic coo.

$\frac{\partial L}{\partial \dot x}$ → conj. momentum with respect to x.

$\Rightarrow \frac{\partial L}{\partial \dot x}$ is a first integral!

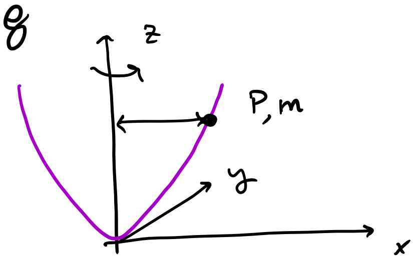
$$\frac{\partial L}{\partial \dot x} = 2m\dot x + m\ell\cos\theta\,\dot\theta$$ is a conserved quantity.

↓ corresponds to the x-coordinate of the quantity of motion of the system.

$$m\vec v_P + m\vec v_Q = m(\dot x, 0) + m(\dot x + \ell\cos\theta\,\dot\theta, \dots)$$

EX2

$\downarrow \vec{g}$



$\vec{\omega} = \omega \hat{z}$

$y = x^2/2$

Lagr. coo. x

FIRST WAY In the fixed frame.

$\vec{OP} = (x\cos(\omega t),\ x\sin(\omega t),\ x^2/2)$

$\vec{v}_p = (\dot{x}\cos(\omega t) - x\omega\sin(\omega t),\ \dot{x}\sin(\omega t) + x\omega\cos(\omega t),\ x\dot{x})$

$|\vec{v}_p|^2 = (1+x^2)\dot{x}^2 + \omega^2 x^2$

$K = \frac{1}{2} m \left[(1+x^2)\dot{x}^2 + \omega^2 x^2 \right]$

$V_{gr} = mg\frac{x^2}{2}$

$L = K - V_{gr}$

SECOND WAY In the rotating frame.

$\vec{OP} = (x,\ x^2/2)$

$\vec{v}_p = (\dot{x},\ x\dot{x}) \Rightarrow |\vec{v}_p|^2 = \dot{x}^2 + x^2\dot{x}^2$

$= (1+x^2)\dot{x}^2$

$$V = V_{gr} + V_{cf} = mg\frac{x^2}{2} - \frac{m\omega^2}{2}x^2$$

$$L = K - V_{gr} - V_{cf}$$

$$= \frac{1}{2}m\dot{x}^2(1+x^2) - \frac{mg}{2}x^2 + \frac{m\omega^2}{2}x^2$$

$$\boxed{\text{Normal form of Lagrange eqs}}$$

Potential energy $V = V(q_1 - q_m)$

$$L(q,\dot{q}) = \frac{1}{2}\sum_{J,K=1}^{m} a_{JK}(q)\dot{q}_J\dot{q}_K - V(q) =$$

$\downarrow$

of MECHANICAL TYPE

$$= K(q,\dot{q}) - V(q).$$

The corresponding Lagr. eq. for q_h coordinate is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = 0$$ That is

$$\frac{d}{dt}\left[\sum_{K=1}^{m} a_{hK}(q)\dot{q}_K\right] - \frac{1}{2}\sum_{J,K=1}^{m}\frac{\partial a_{JK}}{\partial q_h}(q)\dot{q}_J\dot{q}_K + \frac{\partial V}{\partial q_h} = 0$$

$$\Leftrightarrow \sum_{J,K=1}^{m}\frac{\partial a_{hK}}{\partial q_J}\dot{q}_J\dot{q}_K + \sum_{K=1}^{m} a_{hK}\ddot{q}_K -$$

$$-\frac{1}{2}\sum_{J,K=1}^{n} \frac{\partial a_{JK}}{\partial q_h}\dot q_J \dot q_K + \frac{\partial V}{\partial q_h} = 0$$

$\Leftrightarrow$

$$\sum_{K=1}^{n} a_{hK}\ddot q_K = \underbrace{\sum_{J,K=1}^{n}\left[\frac{1}{2}\frac{\partial a_{JK}}{\partial q_h} - \frac{\partial a_{hK}}{\partial q_J}\right]\dot q_J \dot q_K}_{= g_h} \; - \frac{\partial V}{\partial q_h}$$

$$-\frac{\partial V}{\partial q_h} = Q_h$$

In a compact way:

$$a(q)\,\ddot q = Q + g$$

$\downarrow$ Kinetic energy matrix (symmetric & positive definite $\Rightarrow$ invertible !!)

$$g_h = \sum_{J,K=1}^{n}\left[\frac{1}{2}\frac{\partial a_{JK}}{\partial q_h} - \frac{\partial a_{hK}}{\partial q_J}\right]\dot q_J \dot q_K$$

Then ($a(q)$ invertible) :

$$\ddot q = a^{-1}(q)\left[Q + g\right]$$

Normal form of Lagr. eqs.

$$\begin{cases} \dot q = v \\ \dot v (= \ddot q) = a^{-1}(q)\left[Q + g\right] \end{cases}$$

↘ we can apply Cauchy Theo, assuring existence and uniqueness of solutions.

EQUILIBRIA & STABILITY for mechanical systems : $L = \frac{1}{2} \sum_{j,k=1}^{n} a_{jk}(q) \dot{q}_j \dot{q}_k - V(q)$.

↓

$$\ddot{q} = a^{-1}(q) [Q + g]$$

where

$$Q_h = -\frac{\partial V}{\partial q_h}, \quad g_h = \sum_{j,k=1}^{n} \left(-\frac{\partial a_{hk}}{\partial q_j} + \frac{1}{2} \frac{\partial a_{jk}}{\partial q_h} \right) \dot{q}_j \dot{q}_k$$

↓

$$\begin{cases} \dot{q} = v \\ \dot{v} = a^{-1}(q) [Q + g] \end{cases}$$

Equilibria are $(q^*, 0)$ such that

$$\nabla V(q^*) = 0$$

In fact :
$$\begin{cases} \dot{q} = 0 \\ \dot{v} = a^{-1}(q^*) [\underbrace{-\nabla V(q^*)}_{=0} + 0] \end{cases}$$

$\dot{q} = 0$

<u>Stability?</u> As in dim = 1 :

IF $V(q)$ has a strict minimum in q^* then

$(q^*, 0)$ is stable $(\forall\, t \in \mathbb{R})$.

PROOF

$E = K(q, \dot{q}) + V(q) - V(q^*)$ as Lyapunov function.

E has a strict minimum in $(q^*, 0)$ by hypothesis.

Moreover E is a first integral for this conservative system $\Rightarrow$ E is a Lyapun function.

$(L_X E \equiv 0) \Rightarrow (x^*, 0)$ is STABLE. $\boxtimes$

Dim 2 (see first ex. of today ...)

$$\text{Hess}\, V(q^*) = \frac{\partial^2 V}{\partial q_h \partial q_k}(q^*).$$

is positive def. $\Leftrightarrow$ the two eigenvalues are positive. (strictly)

- $\lambda_1, \lambda_2 > 0 \rightarrow (q^*, 0)$ stable.
- λ_1 OR $\lambda_2 = 0 \rightarrow$ we cannot conclude.
- λ_1 OR $\lambda_2 < 0 \rightarrow (q^*, 0)$ unstable.