Exercise 1. Let
\[ V_k(x) = \frac{k}{3}x^3 - x, \quad x \in \mathbb{R} \text{ and } k \in \mathbb{R}. \]

(a) Draw the bifurcation diagram for \( \dot{x} = V_k(x) \).
(b) Draw the cobweb plot for the discrete dynamical system given by the iteration of the map \( V_1(x) \). Determine equilibria and their (linear) (un)stability.
(c) Draw the phase-portraits for \( \ddot{x} = -V_k'(x) \) corresponding to \( k > 0 \), \( k < 0 \) and \( k = 0 \).
(d) Linearize \( \ddot{x} = -V_k'(x) \) around \((\pm 1, 0)\). Establish the quality of these equilibria for the linearized system; say –if it makes sense– the winding direction.
(e) Establish for which values of \( v \in \mathbb{R} \) the solution of \( \ddot{x} = -V_1'(x) \) with initial datum \((1, v)\) is periodic.
(f) How many orbits of \( \ddot{x} = -V_1'(x) \) correspond to the energy value \( E = V_1(-1) \)?

Exercise 2. Let consider the vector field on \( \mathbb{R}^3 \):
\[ X(x, y, z) = \begin{pmatrix} yz^2 \\
-(x-1)^2xz^2 \\
(2-x)x^2yz \end{pmatrix} \]

(a) Prove that \( F(x, y, z) = x^2 + y^2 - z^2 \) is a first integral for \( X \).
(b) Give the definition of (topological) stability and asymptotic stability. Explain why the equilibrium \((1, 0, 2)\) for \( X \) cannot be asymptotically stable.
(c) Can the dynamics corresponding to such \( X \) have a limit cycle?

Exercise 3. Give the definition of invariant set for a continuous flow \( \varphi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Let \( \beta \in \mathbb{R} \) be fixed. For the vector field on \( \mathbb{R}^2 \):
\[ X(x, y) = \begin{pmatrix} -3x + y \\
(\beta - 3)y \end{pmatrix} \]
show that the line \( y = \beta x \) is an invariant set.

Exercise 4. Sketch a phase portrait in the plane consistent with the following information: three equilibria, one saddle and two stable nodes.