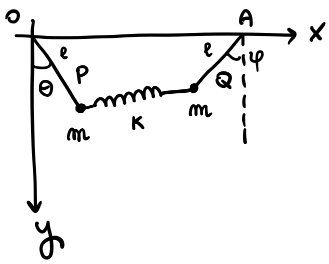
Lesson 20 - 14/11/2022

- **EX 1**




$A = (d, 0)$

Lagrangian (also with gravitational potential)
↳ supporting vertical plane.

$K = \frac{1}{2} m\ell^2 (\dot\theta^2 + \dot\varphi^2)$

$|PQ|^2 = |d - \ell\sin\theta - \ell\sin\varphi,\ \ell\cos\theta - \ell\cos\varphi|^2$

$= d^2 + \ell^2\sin^2\theta + \ell^2\sin^2\varphi - 2\ell d\sin\theta - 2\ell d\sin\varphi +$
$+ 2\ell^2\sin\theta\sin\varphi + \ell^2\cos^2\theta + \ell^2\cos^2\varphi - 2\ell^2\cos\theta\cos\varphi$

$= d^2 + 2\ell^2 - \underbrace{2\ell^2\cos(\theta+\varphi) - 2\ell d(\sin\theta + \sin\varphi)}$ up to constants

$\frac{1}{2}K|PQ|^2$

$V_{el} = -K\ell[\ell\cos(\theta+\varphi) + d(\sin\theta + \sin\varphi)]$

$V_{gr} = -mg\ell\cos\theta - mg\ell\cos\varphi$

($V_{gr} = mgy$ if $y\uparrow$
$V_{gr} = -mgy$ if $y\downarrow$)

$\Rightarrow L = K - V_{el} - V_{gr}$

$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0 \\ \frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} - \frac{\partial L}{\partial\varphi} = 0 \end{cases}$

- Complete proof of Lagrange eqs.
- Let $L = L(q, \dot q, t)$. Then the function:

$E(q,\dot q,t) = \sum_{h=1}^{m} \dot q_h \boxed{\frac{\partial L}{\partial \dot q_h}(q,\dot q,t)} - L(q,\dot q,t)$ (Jacobi integral)

(less 19)

conjugate momentum.

is a first integral iff $\frac{\partial L}{\partial t} = 0$

Proof

$L_X E = \sum_{h=1}^{m}\left[\ddot q_h \cancel{\frac{\partial L}{\partial \dot q_h}(q,\dot q,t)} + \dot q_h \cancel{\frac{d}{dt}\frac{\partial L}{\partial \dot q_h}(q,\dot q,t)}\right] - \sum_{h=1}^{m}\left[\dot q_h \cancel{\frac{\partial L}{\partial q_h}} + \ddot q_h \cancel{\frac{\partial L}{\partial \dot q_h}}\right] - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$

$= \frac{\partial L}{\partial q_h}$ by Lagrange eqs.

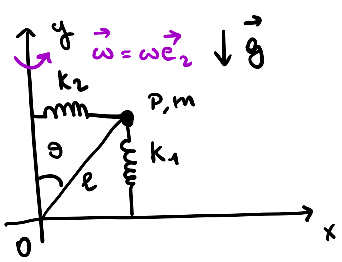
$\frac{d}{dt}\frac{\partial L}{\partial \dot q_h} - \frac{\partial L}{\partial q_h} = 0$

- If $L = K - V = \frac{1}{2}\sum_{h,k=1}^{m} a_{h,k}(q)\dot q_h \dot q_k - V(q)$

Then $E(q,\dot q,t) = \sum_{h,k=1}^{m} \dot q_h a_{h,k}\dot q_k - \frac{1}{2}\sum_{h,k=1}^{m} a_{h,k}\dot q_h \dot q_k + V(q)$

$= K + V$ = total energy of the system.

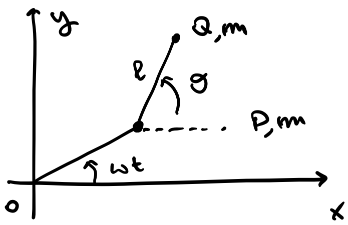
- **EX 2**




P,m

- Lagrangian
- Equilibria
- Their stability
- First integral

- **EX 3**

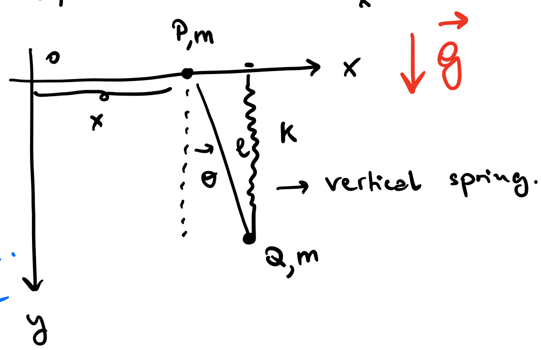



On the plane $Oxy$

- Lagrangian
- Is the total energy conserved?

- **EX 4**

For you, at home.




- Lagrangian
- Equilibria and their stability
- First integral(s).

## Proof of Lagrange eqs

[ System of $N$ points of masses $m_1 - m_N$, $q_1 - q_n$ Lagrangian coordinates. $\vec{F}_i$ $N$ forces, ideal constraints. Then:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} = Q_h \qquad h = 1 - n$$

$K$ total kinetic energy

Recall that $Q_h := \sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h}$ ]

We need to prove that

$$\sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_h} - \frac{\partial K}{\partial q_h} \qquad \forall\, h = 1 - n.$$

Recall that $K = \sum_{i=1}^{N} K_i$, $K_i(\vec{q}, \dot{\vec{q}}, t)$ = kinetic energy of the point $P_i$.

Since we have $\sum_{i=1}^{N}$ in both members, it is sufficient to verify that

$$m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} = \frac{d}{dt}\frac{\partial K_i}{\partial \dot{q}_h} - \frac{\partial K_i}{\partial q_h} \qquad \forall\, h = 1 - n, \quad \forall\, i = 1 - N$$

$$m_i \vec{a}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h} = m_i \frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial \vec{OP}_i}{\partial q_h}\right) - m_i \vec{v}_i \cdot \frac{d}{dt}\frac{\partial \vec{OP}_i}{\partial q_h} =$$

$$= m_i \frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_h}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_h} =$$

$$\frac{\partial \vec{OP}_i}{\partial q_h} = "\frac{\partial \vec{OP}_i}{\partial t} \cdot \frac{\partial t}{\partial q_h}" = \frac{\partial \vec{v}_i}{\partial \dot{q}_h} \quad \text{AND} \quad \frac{d}{dt}\frac{\partial \vec{OP}_i}{\partial q_h} = \frac{\partial \vec{v}_i}{\partial q_h}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_h}\left(\frac{1}{2} m_i |\vec{v}_i|^2\right) - \frac{\partial}{\partial q_h}\left(\frac{1}{2} m_i |\vec{v}_i|^2\right) =$$

$$= \frac{d}{dt}\frac{\partial K_i}{\partial \dot{q}_h} - \frac{\partial K_i}{\partial q_h} \qquad \square$$

## Equalities

- Recall that $\vec{v}_i = \sum_{i=1}^{m} \frac{\partial \vec{OP}_i}{\partial q_h} \dot{q}_h + \frac{\partial \vec{OP}_i}{\partial t} \Rightarrow$

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_h} = \frac{\partial \vec{OP}_i}{\partial q_h}$$

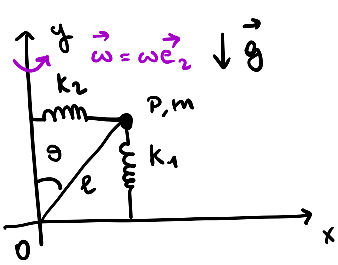
- $f = f(\vec{q}, t)$ [ At the end, we apply the result on $\vec{OP}_i = \vec{OP}_i(\vec{q}, t)$ ]

$$\frac{d}{dt}\frac{\partial f}{\partial q_h} = \sum_{k=1}^{m} \frac{\partial}{\partial q_k}\frac{\partial f}{\partial q_h} \dot{q}_k + \frac{\partial}{\partial t}\frac{\partial f}{\partial q_h} =$$

$$= \frac{\partial}{\partial q_h}\left(\sum_{k=1}^{m} \frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial q_h}\frac{df}{dt}$$

$\Rightarrow$ we obtain (apply to $\vec{OP}_i$ ...) $\quad \frac{\partial}{\partial q_h}\underbrace{\frac{d\vec{OP}_i}{dt}}_{\vec{v}_i} = \frac{\partial \vec{v}_i}{\partial q_h} = \frac{d}{dt}\frac{\partial \vec{OP}_i}{\partial q_h}$

## EX 2




$$\vec{OP} = (\ell \sin\theta, \ell\cos\theta)$$

$$\vec{v}_P = \ell(\dot\theta \cos\theta, -\dot\theta \sin\theta)$$

$$|\vec{v}_P|^2 = \ell^2 \dot\theta^2$$

$$\Rightarrow K = \frac{1}{2} m \ell^2 \dot\theta^2$$

$$V = V_{el} + V_{gr} + V_{cf}$$

$$\underline{V_{el}} = \frac{1}{2} k_1 (\ell\cos\theta)^2 + \frac{1}{2} k_2 (\ell\sin\theta)^2 =$$

$$= \frac{1}{2} k_1 \ell^2 \cos^2\theta + \frac{1}{2} k_2 \ell^2 \sin^2\theta$$

$$\underline{V_{gr}} = mg\,\ell\cos\theta$$

$$\underline{V_{cf}} = -\frac{1}{2} m\omega^2 (\ell\sin\theta)^2 = -\frac{1}{2} m\omega^2 \ell^2 \sin^2\theta$$

$$\Rightarrow L = K - V = \frac{1}{2} m \ell^2 \dot\theta^2 - \frac{1}{2} k_1 \ell^2 \cos^2\theta - \frac{1}{2} k_2 \ell^2 \sin^2\theta -$$

$$- mg\ell\cos\theta + \frac{1}{2} m\omega^2 \ell^2 \sin^2\theta.$$

$\Rightarrow \dots \Rightarrow$ Lagr. eqs.

This is a 1-dim. conservative system $\Rightarrow$ critical points of $V$ are equilibrium configurations $\theta^*$ $[(\theta^*, 0)]$

$$V_\theta = \frac{\partial V}{\partial \theta} = -mg\ell\sin\theta + \ell^2\big(k_1\cos\theta(-\sin\theta) + k_2\sin\theta\cos\theta -$$

$$- m\omega^2\sin\theta\cos\theta\big) =$$

$$= \sin\theta\Big[\underbrace{-mg\ell + \ell^2\left(-k_1\cos\theta + k_2\cos\theta - m\omega^2\cos\theta\right)}_{=0}\Big]$$

$$= 0 \;\to\; \theta = 0, \quad \theta = \pi$$

And - eventually - another pair of equilibria

$$mg\cancel{\ell} = \ell^{\cancel{2}}\left(-k_1 + k_2 - m\omega^2\right)\cos\theta$$

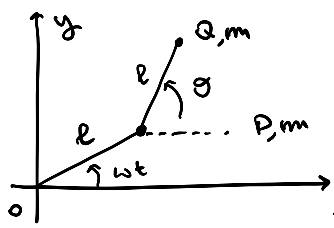
$$\cos\vartheta = \frac{mg}{\ell(k_2 - k_1 - m\omega^2)} \rightarrow \begin{array}{l}\vartheta_3 = \arccos(\dots)\\ \vartheta_4 = -\vartheta_3\end{array}$$

IF $\in (-1, 1)$

Stability depends on sign $V_{\vartheta\vartheta}$ ....

First integral : $E = K + V$ ( first integral ).

EX 3




HORIZONTAL PLANE

- Lagrangian
- Is the total energy conserved?

No because $\frac{\partial L}{\partial t} \neq 0$ !!

(L does depend on time : the constraint is not fixed!)

$$\vec{OP} = (\ell\cos(\omega t), \ell\sin(\omega t))$$

$$\vec{v}_P = (-\ell\omega\sin(\omega t), \ell\omega\cos(\omega t))$$

$$\Rightarrow K_P = \frac{1}{2} m \ell^2 \omega^2$$

$$\vec{OQ} = (\ell\cos(\omega t) + \ell\cos\vartheta, \ell\sin(\omega t) + \ell\sin\vartheta)$$

$$\vec{v}_Q = (-\ell\omega\sin(\omega t) - \ell\dot\vartheta\sin\vartheta, \ell\omega\cos(\omega t) + \ell\dot\vartheta\cos\vartheta)$$

$$|\vec{v}_Q|^2 = (\ell^2\omega^2\sin^2(\omega t) + \ell^2\dot\vartheta^2\sin^2\vartheta + 2\ell^2\omega\dot\vartheta\sin(\omega t)\sin\vartheta + \ell^2\omega^2\cos^2(\omega t) + \ell^2\dot\vartheta^2\cos^2\vartheta + 2\ell^2\omega\dot\vartheta\cos(\omega t)\cos\vartheta)$$

$$K = K_P + \frac{1}{2}m(\ell^2\omega^2 + \ell^2\dot\vartheta^2 + 2\ell^2\omega\dot\vartheta\cos(\vartheta - \omega t))$$

$K_P$ = const

$$= \frac{1}{2} m\ell^2\dot\vartheta^2 + m\ell^2\omega\dot\vartheta\cos(\vartheta - \omega t) \quad (= L)$$

↳ up to constants.

No external forces!