§ Applicazioni lineari

funzione / mappa

$f: V \longrightarrow W$ $\quad V, W$ sp. vett. ~~tra~~ sullo stesso campo K

applicaz (tra insiemi) è detta lineare (K-lineare)

se valgono a) $f(v_1 +_V v_2) = f(v_1) +_W f(v_2) \quad \forall v_1, v_2 \in V$

rispetta la somma

b) $f(\alpha \cdot_V v) = \alpha \cdot_W f(v) \quad \forall v \in V \; e \; \forall \alpha \in K$

rispetta il ·

Esempi

- $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 2x$ $\quad \mathbb{R}$ come sp. vett. su se stesso

  è lineare

  a) $\underbrace{f(x_1+x_2)}_{2(x_1+x_2)} \overset{?}{=} f(x_1) + f(x_2)$

  $2(x_1+x_2) = 2x_1 + 2x_2$ ✓

  b) $f(\alpha x) \overset{?}{=} \alpha f(x)$

  $2(\alpha x) \quad \alpha(2x)$ ✓

- $f: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad x \longmapsto x^2$ $\quad$ Non è lineare!

  a) $f(x_1 + x_2) \overset{?}{=} f(x_1) + f(x_2)$

  $(x_1+x_2)^2 \overset{?}{=} x_1^2 + x_2^2$ $\quad$ NO

- $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

  $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x + y + 2$ $\quad 0$

  a) $f\left[\underbrace{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}}_{\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}}\right] \overset{?}{=} \underbrace{f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}}_{x_1+y_1+2 \; + \; x_2+y_2+2}$ $\quad 0 \quad 0$

  $(x_1+x_2)+(y_1+y_2)+2$ $\overset{?}{\neq}$ NO

  0 SI

- $f: \mathbb{C} \longrightarrow \mathbb{C}$ $\quad \mathbb{C}$ come $\mathbb{R}$-spazio vett

  $z \longmapsto \bar{z}$

  $a+ib \longmapsto a-ib$

a) $\underbrace{f(z_1+z_2)}_{\overline{z_1+z_2}} \overset{?}{=} f(z_1)+f(z_2)$

$\underset{\text{sì}}{=}$ $\overline{z_1} + \overline{z_2}$

b) $f(\alpha z) \overset{?}{=} \alpha f(z)$ $\quad \alpha \in \mathbb{R}$

$\overline{\alpha z} \overset{?}{=} \alpha \overline{z}$ $\quad$ sì

$\overline{\alpha}\,\overline{z}$

NB $\overline{(\ \ )}$ è $\mathbb{R}$-lineare ma non $\mathbb{C}$-lineare.

- $f: V \xrightarrow{\sim} K^n$ $\quad \dim V = n$, $\mathcal{V} = \{v_1, \dots, v_n\}$ base fissata

$v \longmapsto$ coordinate di v risp. a $\mathcal{V}$ $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \rightsquigarrow v = \sum_{i=1}^{m} a_i v_i$

a) $f(v_1+v_2) \overset{\text{sì}}{=} f(v_1)+f(v_2)$ $\quad v_1 = \sum a_i v_i$, $v_2 = \sum b_i v_i$

$v_1+v_2 = \sum_{i=1}^n a_i v_i + \sum_{i=1}^n b_i v_i = \sum_{i=1}^n (a_i+b_i) v_i$

b) $f(\alpha v) \overset{?}{=} \alpha f(v)$ sì $\quad \alpha v = \alpha \sum_{i=1}^m a_i v_i = \sum_{i=1}^m \alpha(a_i v_i)$

$= \sum_{i=1}^n (\alpha a_i) v_i$

- $f: K^n \longrightarrow K^m$ $\quad A \in M_{m,n}(K)$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x \longmapsto Ax$ $\quad (m, \cancel{n)\cdot(n},1)$ ok

prodotto riga per colonna.

$f(x+y) \overset{?}{=} f(x)+f(y)$

Osservo che se $A = (C_1, C_2, \dots C_n)$ con $C_i \in K^m$ colonna i-esima

$Ax = \sum_{i=1}^n x_i C_i$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

a) $A(x+y) \overset{?}{=} Ax + Ay$

$\sum_{i=1}^m (x_i+y_i) C_i \overset{?}{=} \sum x_i C_i + \sum y_i C_i$ $\quad$ sì

Analogo b) $A(\alpha x) \overset{?}{=} \alpha A x$ $\quad \sum (\alpha x_i) C_i \overset{?}{=} \alpha \sum x_i C_i$ sì