Theorem (Bolzano) \( f: I \rightarrow \mathbb{R} \) interval.

\( x_1, x_2 \in I \) s.t. \( x_1 < x_2 \)

\( f(x_1) < 0 \)

\( f(x_2) > 0 \)

\( \Rightarrow \exists \ x \in ]x_1, x_2[ \) s.t.

\( f(x) = 0 \)

Corollary (Intermediate value theorem)

\( f: I \rightarrow \mathbb{R} \) continuous

\( x_1, x_2 \in I \)

\( x < x_2 \)

\( f(x_1) = \alpha \)

\( f(x_2) = \beta \)

\( \alpha < \beta \). If \( x \in ]\alpha, \beta[ \)

\( f(x) = 0 \)

Diagram:

- \( x_1 \), \( x_2 \), \( x \)
- \( I \)
- \( \mathbb{R} \)
Then $\exists x \text{ s.t. } f(x) = x$

**Proof:** $g: I \rightarrow \mathbb{R}$
\[ g(x) = f(x) - x \]
\[ g(x) = f(x) - x = a - x < 0 \]
\[ g(x_2) = f(x_2) - x_2 > b - x > 0 \]
by Bolzano th.
\[ \exists x \text{ s.t. } g(x) = 0 \]
\[ 0 = g(x) = f(x) - x \]
\[ \iff f(x) = x \]
q.e.d.

**Corollary.** $f: [a, b] \rightarrow \mathbb{R}$
continuous. Then
\[ f([a, b]) = [m, M] \]
where $m = \min f$ and $M = \max f$. 

Exercise: \( f: [a, b] \to \mathbb{R} \)

if injective and continuous

Let \( x \in \mathbb{R}, x \neq a, b \). Then

\( f(a) < f(x) < f(b) \)

if \( f(x) \neq f(a) \) then by injectivity

\( f(x) \neq f(b) \)

\( f(a) < f(x) < f(b) \)

\[ \exists x \in \mathbb{R} \text{ s.t. } f(x) = x \]

\[ f'(a) = f'(b) \]
Exercise \( f : \mathbb{R} \to \mathbb{R} \) continuous.

\[
\lim_{x \to -\infty} f(x) = +\infty \\
\lim_{x \to +\infty} f(x) = +\infty
\]

Then there exists a minimum.

By contradiction there is no minimum.

For every \( n \in \mathbb{N} \)

\[ \exists x_0 \in \mathbb{R} \text{ s.t. } f(x_0) < -n \]
\[ \inf f(\mathbb{R}) \geq C \]

\[ C \in \mathbb{R} \quad \text{then } n \in \mathbb{N} \]

\[ \exists \ y_n \in f(\mathbb{R}) \ s.t. \]

\[ C \leq y_n < \frac{C + 1}{n} \]

\[ f(x_n) \text{ is unbounded} \]

\[ \text{there is a subsequence} \]

\[ x_{n_k} \to +\infty \]

\[ \lim_{n \to \infty} f(x_{n_k}) = +\infty \]

\[ y_{n_k} \to c \in \mathbb{R} \]

\[ \inf f(\mathbb{R}) (C) = -\infty \]
\( f(x_n) < -n \)

if \( (x_n) \) is unbounded

\( \exists x_{n_k} \rightarrow +\infty \) (or \(-\infty\))

\( \lim_{k \rightarrow +\infty} f(x_{n_k}) = +\infty \)

by hypothesis \( \lim_{n \rightarrow \infty} f(x_n) = +\infty \)

Finish the exercise with showing that also \( (x_n) \) bounded

is a contradiction (Weierstras Th)
Exercise: \( f, g : \mathbb{R} \to \mathbb{R} \)

1. Find \( f \) and \( g \) s.t.
   - \( f \) is continuous
   - \( g \) is not continuous
   - \( g \circ f \) is continuous

2. Find \( f \) and \( g \) both not continuous, but \( g \circ f \) is continuous

Exercise: \( f(x) = 3x^3 - 8x^2 + x + 3 \)

Show that there are 3 distinct solutions of \( f(x) = 0 \) with \( x, x_1, x_2, x_3 \in \mathbb{R} \), with \( x, x_1 \in ]-\infty, 0[^2 \), \( x_2 \in ]0, 1[^2 \), \( x_3 \in ]1 + \infty[^2 \).
\[ f(0) = 3 \]
\[ f(-1) = -3 - 8 - 1 + 3 = -9 \]
\[ f(0) > 0 \quad f(-1) < 0 \]
\[ \therefore x_1 \in ]-1, 0[ \]

By Intermediate Value Theorem

\[ f(x_2) = 0 \]
\[ f(1) = 3 - 8 + 1 + 3 = \]
\[ = -1. \]

\[ \exists \ x_2 \in ]0, 1[ \]

s.t.

\[ f(x_2) = 0 \]

\[ f(0) = 24 - 32 + 2 + 3 = 0 \]
\[ = -8 + 2 + 3 = -3 \leq 0 \]
\[ f(3) = 243 - 72 + 3 + 3 > 0 \]
\[ \Rightarrow \quad x_3 \in ]1, 3[ \]
\[ \therefore \quad p. d. \]

Exercise

\[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]

\[ a_n \neq 0 \]

a polynomial of degree \( n \), \( n \) odd.

Show that the equation \( p(x) = 0 \) has (at least) one solution.
Suppose \( a_n > 0 \)

\[
\lim_{x \to \infty} a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 =
\]

\[
= \lim_{x \to \infty} x^n \left( a_n + a_{n-1} \frac{x}{x} + \ldots + \frac{a_0}{x^n} \right)
\]

\[
= +\infty
\]

If \( a_n < 0 \)

\[
\lim_{x \to \infty} p(x) = -\infty
\]

If \( a_n > 0 \)

\[
\lim_{x \to -\infty} p(x) = -\infty
\]

\[
a_n < 0 \quad \lim_{x \to -\infty} p(x) = +\infty
\]

If \( d_n > 0 \)

Since \( p(x) \xrightarrow[x \to \infty]{} +\infty \)

There is \( M \) s.t. \( \forall x > M \)

\[
p(x) > 1
\]
Since $p(x) \to -\infty$ as $x \to \pm \infty$,

there is $K$ s.t.

$\forall x < K \quad p(x) < -1$

Choose $x_1$, according to $\textcircled{a}$ and $x_2$

according to $\textcircled{b}$

$p(x_1) > 1 > 0$

$p(x_2) < -1 < 0$

Bolzmann distribution $\overline{\xi}$ in the interval of extremes such that $p(\overline{\xi}) = 0$
Author method: I know I can factor

\[ p(x) = p_1(x) \cdot p_2(x) \cdots p_k(x) \]

due to all \( p_i \) being of degree 1 or 2.

Since \( p \) has odd degree there must be at least one \( p_i \) having degree 1.

\[ p_i = (ax + b) \]

\[ p(x) = (ax + b)(q(x)) \]

If \( x = -\frac{b}{a} \)

\[ p(x) = 0 \cdot q(x) = 0 \]
$f: D \rightarrow \mathbb{R}$

Simplest functions are the linear ones, that
depicted by:

\[ r(x) = mx + q, \]

where graphs are lines

\[ x = \overline{x} \]

The line $x = \overline{x}$ is a vertical asymptote
of $f$ if $\lim_{x \to \overline{x}} f(x) = \pm \infty$.
or \( \lim_{x \to \infty} f(x) = \pm \infty \)

If \( \lim_{x \to \infty} f(x) = \ell \in \mathbb{R} \)
we say that \( f(x) \) has the horizontal asymptote \( y = \ell \) for \( x \to \infty \).
\[
\lim_{x \to a^-} f(x) = +\infty \\
(a - \infty)
\]

\[y = mx + q\]

\[
\lim_{x \to +\infty} (f(x) - (mx + q)) = 0 \quad \ast
\]

If \[y = mx + q\] is called asymptote of \(f\) for \(x \to +\infty\).
Determine \( m \) and \( q \).

By \( \circ \)

\[
\lim_{x \to 10} \frac{f(x) - mx - q}{x} = 0
\]

\[
11
\]

\[
\lim_{x \to 10} \frac{f(x) - m}{x} = 0
\]

\[
\Rightarrow \quad m = \lim_{x \to 10} \frac{f(x)}{x}
\]

From \( \circ \)

\[
q = \lim_{x \to 10} \frac{f(x) - mx}{x - 10}
\]
f(x) = x + x^2
\[
\lim_{x \to 2} \frac{x^2 - x^2}{\sqrt{x^3 + x}} = \frac{1}{2}
\]

\[y = x + \frac{1}{2} \]

is the asymptote

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x + \log x}{x} = \lim_{x \to +\infty} \frac{x + \log x}{x} = 1
\]

\[m = 1\]

\[q = \lim_{x \to +\infty} (x - x) = \lim_{x \to +\infty} \log x - x = \lim_{x \to +\infty} \log x = +\infty\]