Lesson 15 - 02/11/2022

- $\begin{cases} \dot{x} = 2x(1+y) - \sin(x+y) \\ \dot{y} = (x-1)y \end{cases}$ winding direction of the stable spiral in $(1,-1)$?

- Phase-portrait of $\ddot{x} = x^3 + x^2 = -V'(x)$
($\Rightarrow$ $V(x) = -\frac{x^3}{4}\left(\frac{4}{3} + x\right)$ (up to constants))

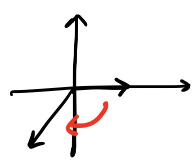
- Let $m = 2$. And
$V(x) = x^2(1-x)(3-x)$
(a) Phase-portrait for $m\ddot{x} = -V'(x)$.
(b) Period for $x(0) = 1$ and $\dot{x}(0) = 0$ (only formula).
(c) Estimate the period (Recall that $\int_a^b \frac{dx}{\sqrt{-(a-x)(b-x)}} = \pi$.

- In the pendulum, the time to reach the unstable position is $+\infty$ (explicit computation).

Constrained dynamical systems $\rightarrow$ From now on, see also Benettin notes on Lagrangian Mechanics on STEM-MOODLE.

$\boxed{1}$ Recall (last lecture) that in $(1,-1)$ there is a stable spiral. Moreover:

$J(1,-1) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$



$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

The rotation is clockwise.

$\boxed{2}$ $\ddot{x} = x^3 + x^2 = -V'(x) \Rightarrow V(x) = -\frac{x^3}{4}\left(\frac{4}{3} + x\right)$

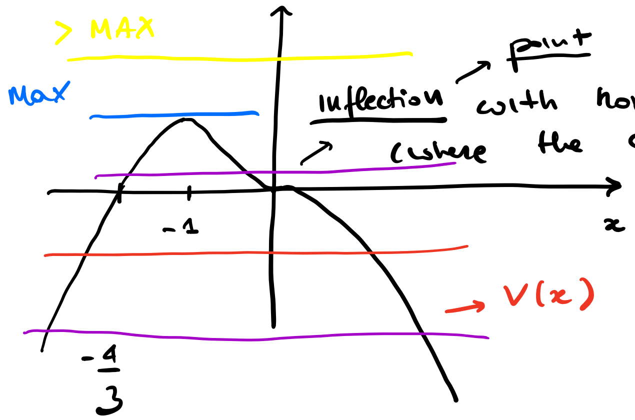
Draw the graph for $V(x)$.

$\lim_{x \to \pm\infty} V(x) = -\infty$.

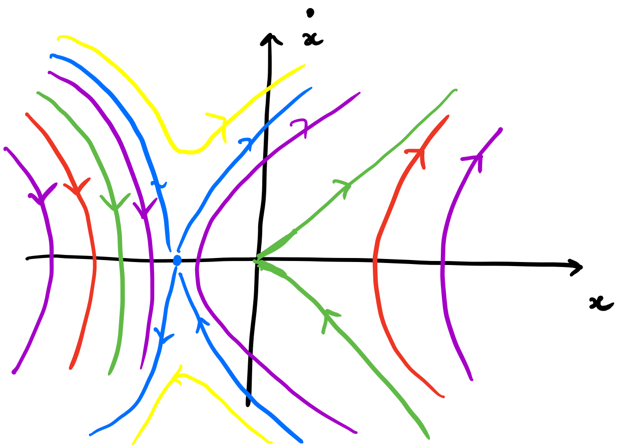
First derivative $V'(x) = -x^3 - x^2 = -x^2(x+1) = 0$

$\Leftrightarrow \begin{cases} x = 0 \\ x = -1 \end{cases}$ OR EQUILIBRIA ARE $(0,0)$ and $(-1,0)$

Intersections with x-axis: $x = 0$ OR $x = -\frac{4}{3}$



inflection point with horizontal tangent (where the concavity changes!)



$\boxed{3}$ $m = 2$

$V(x) = x^2 (1 - x)(3 - x)$

a) $m\ddot{x} = -V'(x)$ phase-portrait

b) Implicit formula for the period of the orbit

$x(0) = 1, \quad \dot{x}(0) = 0$

c) Estimate the period (recall that $\int_a^b \frac{dx}{\sqrt{-(a-x)(b-x)}} = \pi$)

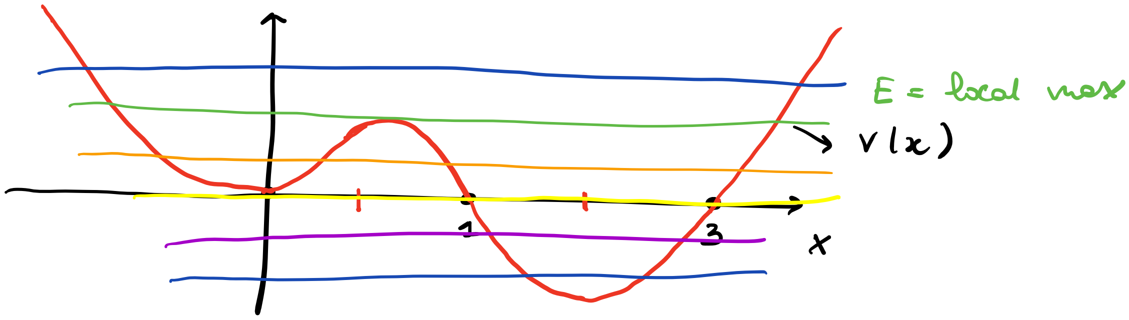
Sol Graph of $V(x)$

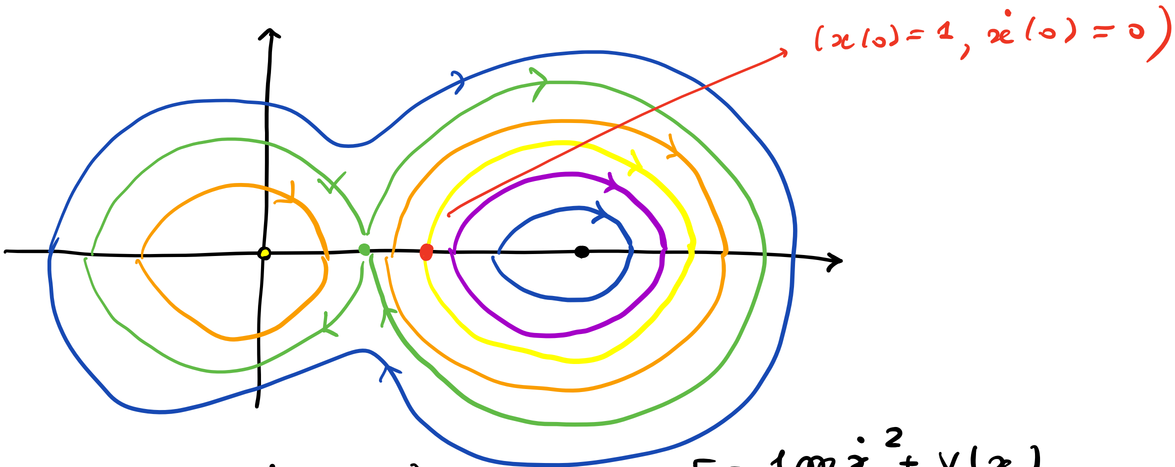
$\lim_{x \to \pm\infty} V(x) = +\infty.$

$V'(x) = \dots = 2x [2x^2 - 6x + 3]$

$V'(x) = 0 \iff x = 0$ OR $x_{12} = \frac{3 \pm \sqrt{3}}{2}$ Both positive.

Intersections with x-axis: $x = 0$, $x = 1$, $x = 3$





$E(x(0)=1,\ \dot{x}(0)=0) = 0 \longrightarrow$ $E = \frac{1}{2} m \dot{x}^2 + V(x)$

$\dot{x} = \frac{dx}{dt}$

So $T = 2\int_1^3 \frac{dx}{\sqrt{\frac{2}{m}[E - V(x)]}}$

$= 2\int_1^3 \frac{1}{x}\frac{dx}{\sqrt{-x^2(1-x)(3-x)}}$ Implicit formula.

$m=2,\ E=0$ $\quad x>0$

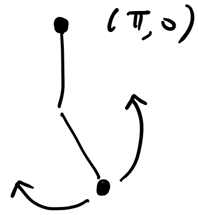
Since $1 \leq x \leq 3 \Rightarrow \boxed{\frac{1}{3} \leq \frac{1}{x} \leq 1}$

So we can estimate the period by substituting 3 and 1 in x^2.

$$\frac{2}{3}\underbrace{\int_1^3 \frac{dx}{\sqrt{-(1-x)(3-x)}}}_{=\pi} \leq T \leq 2\underbrace{\int_1^3 \frac{dx}{\sqrt{(1-x)(3-x)}}}_{=\pi}$$

$$\boxed{\frac{2}{3}\pi \leq T \leq 2\pi}$$

Remark: In the pendulum the time to reach the unstable position is $+\infty$.



$\downarrow$

A consep. of Cauchy uniqueness Theo.

- But we can also do the explicit computation!

$\ddot{x} = -\sin x$

$V(x) = -\cos x$ with max $= 1$ (energy level of the separatrix).

$$\frac{1}{2}\dot{x}^2 - \cos x = 1$$

$$\dot{x}^2 = 2\left[1 + \cos x\right]$$

$$\dot{x} = +\sqrt{2(1+\cos x)} \qquad \frac{dx}{dt} = \sqrt{2}\sqrt{1+\cos x}$$

$\downarrow$

choose $\dot{x}_0 > 0$

$$t = \int_{x_0}^{\pi} \frac{dx}{\sqrt{2}\sqrt{1+\cos x}}$$

$$1+\cos x = (1-1) - \sin x\big|_{x=\pi}(x-\pi) - \frac{1}{2}\cos x\Big|_{x=\pi}(x-\pi)^2 + o(x-\pi)^2$$

$\downarrow$

neigh of π

$$= 0 + 0 + \frac{1}{2}(x-\pi)^2 + \ldots$$

$$\frac{1}{\sqrt{1+\cos x}} \underset{\text{neigh. of } \pi}{\simeq} \frac{1}{\sqrt{\frac{1}{2}(x-\pi)^2}} = \frac{\sqrt{2}}{|x-\pi|}$$

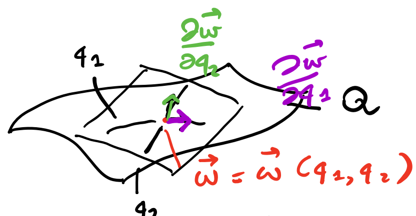
$\Rightarrow$

$$t \simeq \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \int_{x_0}^{\pi} \frac{dx}{|x-\pi|} = +\infty$$

since $1/x$ is not integr. in a neigh of 0.

Constrained dynamical systems → Benettin notes PDF on-line

- Surface $Q \subseteq \mathbb{R}^3$ (dim = 2)



can be described in two ways :

- Implicitly , by $F(x, y, z) = 0$, with $F \in C^\infty(\mathbb{R}^3; \mathbb{R})$

$$\nabla F(x, y, z)\Big|_Q = \begin{pmatrix} \partial_x F \\ \partial_y F \\ \partial_z F \end{pmatrix}(x, y, z)\Big|_Q \neq 0.$$

- By a local parametrization :

$x = x(q_1, q_2)$, $y = y(q_1, q_2)$, $z = z(q_1, q_2)$

with $(q_1, q_2) \in \underbrace{\mathcal{U}}_{\text{open set}} \subseteq \mathbb{R}^2$

$$\vec{w} = \vec{w}(q_1, q_2) = (x, y, z)$$

In particular, Q admits tangent plane at every point $\vec{w}$ and the Jacobian :

$$\frac{d\vec{w}}{dq} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} \end{pmatrix}$$

$$\begin{array}{cc} \| & \| \\ \frac{\partial \vec{w}}{\partial q_1} & \frac{\partial \vec{w}}{\partial q_2} \end{array}$$

has max rk = 2 .

↓

As a consequence, the pair of vectors $\frac{\partial \vec{w}}{\partial q_1}$, $\frac{\partial \vec{w}}{\partial q_2}$

is – in any point $\vec{w}$ of Q – a basis for the local tangent plane.

$$\Rightarrow \underbrace{\delta \vec{w}}_{\in T_{\vec{w}} Q} = \sum_{h=1}^{2} \frac{\partial \vec{w}}{\partial q_h} \delta q_h$$

$$Q : \quad TQ = \bigcup_{\vec{\omega} \in Q} T_{\vec{\omega}} Q \overset{?}{=} Q \times \mathbb{R}^n$$

$$TS^2 \neq S^2 \times \mathbb{R}^2 \quad \text{Not alway true}$$

Examples

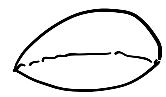
Sphere S^2 $\quad F(x,y,z) = 0$

- $x^2 + y^2 + z^2 - R^2 = 0$

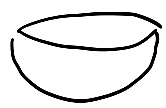
$\nabla F(x,y,z)\big|_{S^2} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0$ is satisfied.

- By a local parameterization.

$x = q_1, \quad y = q_2, \quad z = \sqrt{R^2 - q_1^2 - q_2^2}$



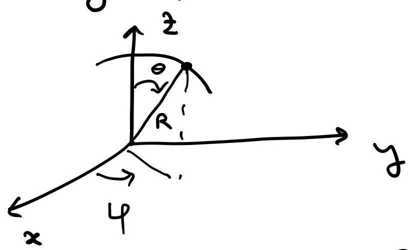
$x = q_2, \quad y = q_2, \quad z = -\sqrt{R^2 - q_1^2 - q_2^2}$



North emisphere, cond. on the rank.

$$\frac{\partial \vec{w}}{\partial \vec{q}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{-\cancel{2} q_1}{\cancel{2}\sqrt{R^2 - q_1^2 - q_2^2}} & \dfrac{-q_2}{\sqrt{R^2 - q_1^2 - q_2^2}} \end{pmatrix} \quad \text{has } rk = 2$$

- By spherical coordinates.



$$\begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases}$$

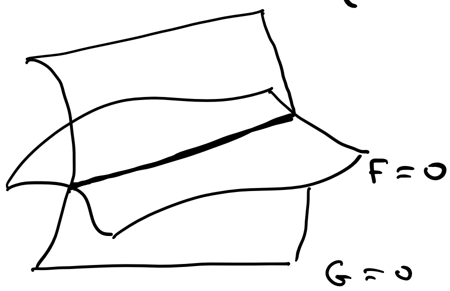
$$\frac{d\vec{\omega}}{d\vec{q}} = \begin{pmatrix} R\cos\theta\cos\varphi & -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi & R\sin\theta\cos\varphi \\ -R\sin\theta & 0 \end{pmatrix}$$

has $rk = 2$ outcircle north and south poles.

- <u>Curve</u> $Q \subseteq \mathbb{R}^3$ (dim 1)

– <u>Implicitly</u>

$$\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$$



$F, G \in C^\infty(\mathbb{R}^3; \mathbb{R})$

and

$$\operatorname{rk} \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{pmatrix} = \max = 2.$$

– By 1-parameter.

$x = x(q_1)$, $y = y(q_1)$, $z(q_1) = z$

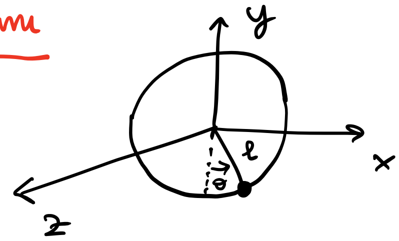
$$\frac{d\vec{w}}{dq_1} = \begin{pmatrix} \partial x / \partial q_1 \\ \partial y / \partial q_1 \\ \partial z / \partial q_1 \end{pmatrix} \neq 0$$

<u>Pendulum</u>

$\boxed{S^1}$

$\downarrow$

<u>circle</u>



$$\begin{cases} x^2 + y^2 - \ell^2 = 0 \\ z = 0 \end{cases}$$

$$\begin{cases} x = \ell \sin\theta \\ y = -\ell\cos\theta \\ z = 0 \end{cases} \rightarrow \frac{d\vec{w}}{d\theta} = \begin{pmatrix} \ell\cos\theta \\ \ell\sin\theta \\ 0 \end{pmatrix}$$

$\neq 0$

—x—x—