Lesson 13 - 26/10/2022

- Lotka-Volterra model (~1930, prey-predator model)
  \[
  \begin{align*}
  \dot{x} &= \alpha x - \beta xy \\
  \dot{y} &= -\gamma y + \delta xy
  \end{align*}
  \]
  \(x = \text{nr of prey}, \ y = \text{nr of predator} \quad \alpha, \beta, \gamma, \delta > 0 \quad x, y > 0\)

- Modified Lotka-Volterra model
  \[
  \begin{align*}
  \dot{x} &= \alpha x - \beta xy - \varepsilon x^2 \\
  \dot{y} &= -\gamma y + \delta xy
  \end{align*}
  \]
  \(\varepsilon > 0 \text{ small}\)

- The limit cycle phenomenon.
- The mechanical clock.
- Toward one-dimensional maps...


\[
\begin{align*}
  &x = \text{nr of prey}, \ y = \text{nr of predator} \\
  \dot{x} &= \alpha x - \beta xy \\
  \dot{y} &= -\gamma y + \delta xy
\end{align*}
\]

We solve qualitatively the system by using a first integral.

\[
\frac{dx}{\alpha x - \beta xy}, \quad \frac{dy}{-\gamma y + \delta xy}
\]

Therefore \(\text{"dt"} = \frac{1}{\alpha x - \beta xy} \, dx = \frac{1}{-\gamma y + \delta xy} \, dy\)

Hence: \((-\gamma y + \delta xy) \, dx - (\alpha x - \beta xy) \, dy = 0\)

Divide by \(xy \ (>0)\), and obtain:

\[
\left( \frac{-\gamma}{x} + \delta \right) \, dx - \left( \frac{\alpha}{y} - \beta \right) \, dy = 0
\]

This first member is the diff. of this function:

\[
F(x, y) = \left( -\gamma \log x + \delta x \right) - \left( \alpha \log y - \beta y \right)
\]

\[
F(x, y) \text{ results a first integral. So, n'ty level}
\]
sets are invariant wrt the dynamics.

\[ \nabla F(x, y) = (0, 0) \] if \( \begin{cases} \frac{-x + \delta}{x} = 0 & \Rightarrow \quad x = \frac{\delta}{8} \\ \frac{-x y + \beta}{y} = 0 & \Rightarrow \quad y = \frac{\alpha}{\beta} \end{cases} \]

A critical point for \( F(x, y) \). We study the Hessian to check if minimum/maximum.

\[ \text{Hess } F(x, y) = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = 0 \]

\[ \text{Hess } F(\frac{\delta}{8}, \frac{\alpha}{\beta}) \] is positive def. \( \Rightarrow \) \( (\frac{\delta}{8}, \frac{\alpha}{\beta}) \) is a (strict) minimum.

By using level sets \( \nabla F(x, y) = c \quad (c > \min F) \) we obtain the contour phase portrait.

2. A correction of previous L-V model.

\[ \begin{cases} \dot{x} = \alpha x - \beta x y - \varepsilon x^2 \\ \dot{y} = -\gamma y + \delta x y \end{cases} \quad \varepsilon > 0 \text{ small} \]

In such a case, we det. and classify equilibria by linearization.
on y-axis: $\dot{y} = -\gamma y$

on x-axis: $\dot{x} = \alpha x - \varepsilon x^2 = x(\alpha - \varepsilon x)$

$(\alpha - \varepsilon x) > 0 \Rightarrow x < \alpha / \varepsilon$

Equilibria:

$\dot{x} = \alpha x - \beta xy - \varepsilon x^2 = 0 \Rightarrow x(\alpha - \beta y - \varepsilon x) = 0$

$\Rightarrow x = 0$ or $\alpha - \beta y - \varepsilon x = 0$

$\dot{y} = -\gamma y + \delta xy = 0 \Rightarrow y(-\gamma + \delta x) = 0$

$\Rightarrow y = 0$ or $x = \sigma / \delta$

$c_1 (0, 0)$

$c_2 \left( \frac{\alpha}{\varepsilon}, 0 \right)$

$c_3 \left( \frac{\sigma}{\delta}, \frac{\alpha}{\beta} - \frac{\varepsilon \sigma}{\delta \beta} \right)$

$J_x(x, y) = \begin{pmatrix} \alpha - \beta y - 2\varepsilon x & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix}$

$|c_1| = (0, 0)$

$J_x(0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix} \Rightarrow \det < 0$

$c_1$ (as expected) is a SADDLE

$|c_2| = \left( \frac{\alpha}{\varepsilon}, 0 \right)$

$J_x \left( \frac{\alpha}{\varepsilon}, 0 \right) = \begin{pmatrix} -\alpha & -\beta \alpha / \varepsilon \\ 0 & -\gamma + \delta \alpha / \varepsilon \end{pmatrix}$
\[ \text{det} = \lambda^2 - \lambda^2 \delta < 0 \Rightarrow C_2 \text{ is a SADDLE.} \]

\[ E \text{ small} \]

\[ [C_3] = \left( \frac{\delta}{\delta}, \frac{\alpha}{\beta} - \frac{E \sigma}{\delta} \frac{\xi}{\beta} \right) \]

\[ J_{x}(C_3) = \begin{pmatrix} \frac{\delta}{\delta} - \frac{E \sigma}{\delta} & -\beta \sigma / \delta \\ \frac{E \sigma}{\beta} & 0 \end{pmatrix} \]

\[ \text{det} = \frac{\beta \sigma}{\delta} \left( \frac{\delta}{\beta} - \frac{E \sigma}{\beta} \right) = \gamma \delta - \frac{E \sigma^2}{\delta} > 0 \]

\[ \text{tr} = -\frac{E \sigma}{\delta} < 0 \]

\[ \Delta = (\text{tr})^2 - 4 \text{det} = \frac{E^2 \delta^2}{\delta^2} - 4 \gamma \sigma + 4 E \sigma^2 \]

\[ < 0 \]

\[ \text{since } E > 0 \text{ small!} \]

Mathematicians and biologists dismissed the previous LV models since realistic models should predict a single closed orbit or perhaps finitely many, but not a continuum of periodic motions (1st case) or an attractor (2nd case).

The Limit Cycle Phenomenon

We intend to construct a mathematical model reproducing the phenomenology of the mechanical clock. It differs from conservative systems (harmonic oscillator, pendulum) since
there is dissipation,
there is a unique periodic motion, of fixed amplitude.

A correct model for a mechanical clock has a unique closed trajectory and the other trajectories approach this one asymptotically.

**LIMIT CYCLE**

This term came from Poincaré (1854 - 1912)

**DEF** A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.

Limit cycles can't occur in linear systems $\dot{x} = Ax$

If $x(t)$ is a period solution of $\dot{x} = Ax$ then also $\alpha x(t)$ ($\alpha \neq 0$) is a period solution of $\dot{x} = Ax$.

First (simple) example

\[
\begin{align*}
\dot{z} &= z(1 - 2) \\
\dot{\theta} &= 1 \\
\dot{z} &\geq 0, \theta \in [0, 2\pi] \\
\dot{z} &= z(1 - 2)
\end{align*}
\]

Polar coordinates on the plane.
**Model of the mechanical clock**

\[
\begin{cases}
\dot{z} = v \\
\dot{v} = -\omega^2 x - 2\mu v
\end{cases}
\]

Eqs. for the harmonic oscillator + dissipative term.  
\( \omega^2 = \frac{K}{m} \)

Take an initial point \((0, v_0)\), \(v_0 > 0\)

\[
\begin{align*}
\ddot{x} &= -\omega^2 x - 2\mu x \\
\dot{x} + 2\mu x + \omega^2 x &= 0
\end{align*}
\]

This eq. can be expl. solved and

\[
x(t) = \frac{v_0}{a} e^{-\mu t} \sin(\sigma t)
\]

where \( \sigma = \sqrt{\omega^2 - \mu^2} \)

\[
v(t) = v_0 e^{-\mu t} \left( -\frac{\mu}{\sigma} \sin(\sigma t) + \cos(\sigma t) \right)
\]

\(x(t)\) crosses the \(v > 0\) axis periodically, with period \(T = 2\pi / \sigma\).

The corresponding velocities are:

\[
\begin{align*}
v_0 &= v_0 \\
v_1 &= v_0 e^{-\frac{2\pi \mu}{\sigma}} = v_0 e^{-\mu T} \\
v_2 &= v_1 e^{-\mu T}
\end{align*}
\]
\[ v_3 = v_2 e^{-\mu T} \ldots \]

That is:
\[
\begin{cases}
    v_{k+1} = a \cdot v_k \\
    v_k = a^k \cdot v_0
\end{cases}
\]

This model - without an external force - is destined to stop ...

So we add an external force as follows:

When the point \( P \) of mass \( m \) passes through \( \sigma > 0 \), it receives a positive impulse which increases the velocity of a fixed quantity.

In formulas:
\[
\begin{align*}
    v_0 \\
    a \cdot v_0 + b &= v_1 \\
    a \cdot v_1 + b &= v_2 \\
    a \cdot v_2 + b &= v_3
\end{align*}
\]

That is
\[
v_{k+1} = a \cdot v_k + b \quad \forall \ k \in \{0, 1, 2, \ldots \}
\]

where \( a = e^{-\mu T} < 1 \), \( b > 0 \) fixed.

Dynamics?!

All solutions approach asymptotically the limit.
cycle (defined periodic solution of fixed
amplitude).

\[ \phi(u^*) = a u^* + b = u^* \Rightarrow u^* = \frac{b}{1-a} > 0 \]

since \( a < 1 \)