Lesson 13 - 26/10/2022

- Lotka-Volterra model (~1930, prey-predator model)

$$\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = -\gamma y + \delta x y \end{cases}$$

(x = nr of prey)
(y = nr of predators)

$\alpha, \beta, \gamma, \delta > 0$

- Modified Lotka-Volterra model

$$\begin{cases} \dot{x} = \alpha x - \beta x y - \varepsilon x^2 \\ \dot{y} = -\gamma y + \delta x y \end{cases}$$

$\varepsilon > 0$ small.

- The limit cycle phenomenon.
- The mechanical clock.
- Towars one-dimensional maps...

1 The "simplest" Lotka-Volterra model.

x = nr of prey, y = nr of predator.

$$\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = -\gamma y + \delta x y \end{cases}$$

ALL PARAMETERS > 0.
$x, y \geq 0$.

We solve qualitatively this system by using a first integral.

$$\frac{dx}{dt} = \alpha x - \beta x y, \quad \frac{dy}{dt} = -\gamma y + \delta x y$$

Therefore "dt" $= \dfrac{1}{\alpha x - \beta x y} dx = \dfrac{1}{-\gamma y + \delta x y} dy$

Hence: $(-\gamma y + \delta x y)\, dx - (\alpha x - \beta x y)\, dy = 0$

Divide by xy (>0), and obtain:

$$\left(\frac{-\gamma}{x} + \delta \right) dx - \left(\frac{\alpha}{y} - \beta \right) dy = 0$$

This first member is the diff. of this function:

$$F(x,y) = (-\gamma \log x + \delta x) - (\alpha \log y - \beta y)$$
$$= -\gamma \log x + \delta x - \alpha \log y + \beta y$$

$F(x,y)$ results a first integral. So, n'ty level sets are invariant wrt the dynamics.

$$\nabla F(x,y) = (0,0) \text{ iff } \begin{cases} -\dfrac{\gamma}{x} + \delta = 0 \longrightarrow x = \gamma/\delta \\ -\alpha/y + \beta = 0 \longrightarrow y = \alpha/\beta \end{cases}$$

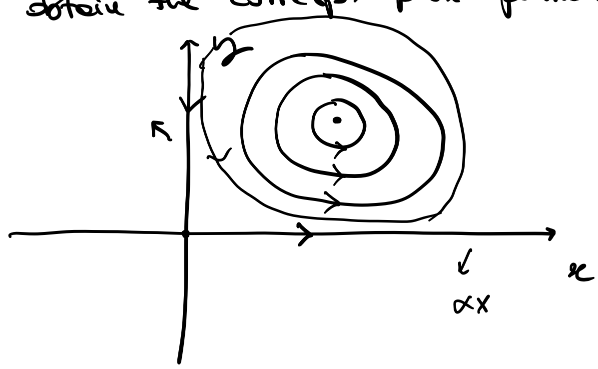
$\exists!$ critical point for $F(x,y)$. We study the Hessian to check if minimum / maximum.

$$\text{Hess } F(x,y) = \begin{pmatrix} \gamma/x^2 & 0 \\ 0 & \alpha/y^2 \end{pmatrix} \Longrightarrow$$

Hess $F(\gamma/\delta, \alpha/\beta)$ is positive def. $\Rightarrow$ $(\gamma/\delta, \alpha/\beta)$ is a (strict) minimum.

$$\begin{pmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x \partial y} \\ \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial y^2} \end{pmatrix}$$

By using level sets of $F(x,y) = c$ $(c \geq \min F)$ we obtain the corresp. phase-portrait.



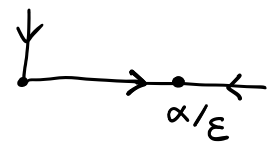
$\boxed{2}$ A correction of previous L-V model.

$\varepsilon > 0$ small

$$\begin{cases} \dot{x} = \alpha x - \beta x y - \varepsilon x^2 \\ \dot{y} = -\gamma y + \delta x y \end{cases}$$

In such a case, we det. and classify equilibria by linearization.

On y-axis : $\dot{y} = -\gamma y$



On x-axis : $\dot{x} = \alpha x - \varepsilon x^2 =$

$= x(\alpha - \varepsilon x)$

$(\alpha - \varepsilon x) > 0 \iff x < \alpha/\varepsilon$

Equilibria :

$\dot{x} = \alpha x - \beta x y - \varepsilon x^2 = 0 \iff x(\alpha - \beta y - \varepsilon x) = 0$

$\Rightarrow x = 0$ OR $\alpha - \beta y - \varepsilon x = 0$

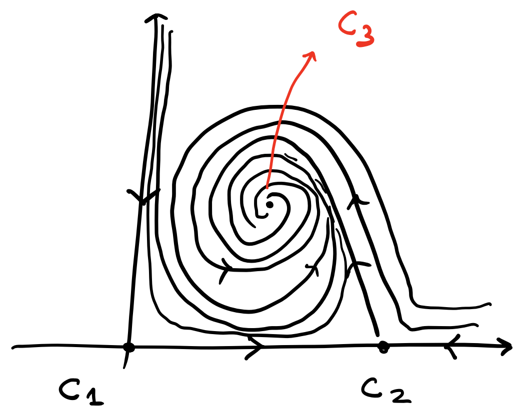
$\dot{y} = -\gamma y + \delta x y = 0 \iff y(-\gamma + \delta x) = 0$

$\Rightarrow y = 0$ OR $x = \gamma/\delta$

$C_1 = (0, 0)$

$C_2 = \left(\frac{\alpha}{\varepsilon}, 0\right)$

$C_3 = \left(\gamma/\delta, \frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\delta\beta}\right)$



$$J_X(x,y) = \begin{pmatrix} \alpha - \beta y - 2\varepsilon x & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix}$$

$\boxed{C_1} = (0,0)$

$$J_X(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \Rightarrow$$

det < 0
C_1 (as expected)
is a SADDLE

$\boxed{C_2} = \left(\frac{\alpha}{\varepsilon}, 0\right)$

$$J_X(\alpha/\varepsilon, 0) = \begin{pmatrix} -\alpha & -\beta\alpha/\varepsilon \\ 0 & -\gamma + \frac{\delta\alpha}{\varepsilon} \end{pmatrix}$$

$$\det = \alpha\gamma - \frac{\alpha^2\delta}{\varepsilon} < 0 \Rightarrow C_2 \text{ is a SADDLE.}$$

$\varepsilon \to$ small

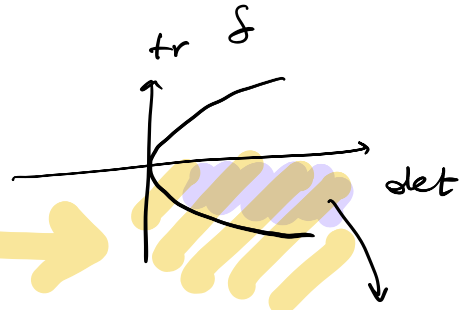
$$\boxed{C_3} = \left(\gamma/\delta ,\ \frac{\alpha}{\beta} - \frac{\varepsilon\gamma}{\delta\beta} \right)$$

$$Jx(C_3) = \begin{pmatrix} \cancel{\alpha} \cancel{-\alpha} - \frac{\varepsilon\gamma}{\delta} & -\beta\gamma/\delta \\ \frac{\delta\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta} & 0 \end{pmatrix}$$

since $\varepsilon > 0$ small.

$$\det = \frac{\beta\gamma}{\delta}\left(\frac{\delta\alpha}{\beta} - \frac{\varepsilon\gamma}{\beta}\right) = \gamma\alpha - \frac{\varepsilon\gamma^2}{\delta} > 0$$

$$tr = -\frac{\varepsilon\gamma}{\delta} < 0$$



C_3 is a stable spiral.

$$\Delta = (tr)^2 - 4\det =$$

$$= \frac{\varepsilon^2\gamma^2}{\delta^2} - 4\gamma\alpha + 4\frac{\varepsilon\gamma^2}{\delta} < 0$$

since $\varepsilon > 0$ small!

→ Mathematicians and biologists dismissed the previous L-V models since realistic models should predict a SINGLE closed orbit or perhaps finitely many, but not a continuous family of periodic motions (1st case) or an attractor (2nd case).

- THE LIMIT CYCLE PHENOMENON.

We intend to construct a mathematical model reproducing the phenomenology of the mechanical clock. It differs from conservative systems (harmonic oscillator, pendulum) since

→ there is dissipation.

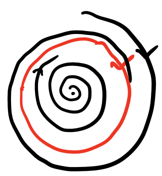
→ there is a unique periodic motion, of fixed amplitude.

A correct model for a mechanical clock has a unique closed trajectory and the other trajectories approach this one asymptotically.

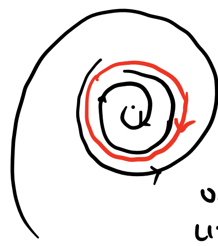
LIMIT CYCLE → This term come from Poincaré (1854 - 1912)

DEF A limit cycle is an isolated closed trajectory.

Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.



STABLE LIMIT CYCLE

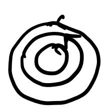


UNSTABLE LIMIT CYCLE.

Limit cycles can't occur in linear systems $\dot{x} = Ax$ ($x \in \mathbb{R}^n$).

If $x(t)$ is a period solution of $\dot{x} = Ax$ then also $cx(t)$ ($c \neq 0$) is a period solution of $\dot{x} = Ax$

$\Downarrow$



we have a 1-parameter family of closed orbits and $x(t)$ cannot be an isolated closed orbit.

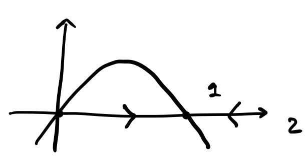
First (simple) example

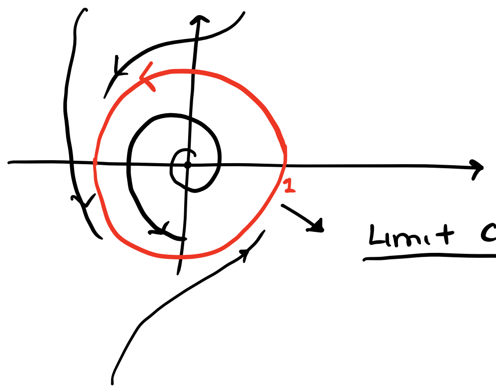
$$\begin{cases} \dot{r} = \underline{r(1-r)} \\ \dot{\theta} = 1 \end{cases}$$

$r \geq 0$, $\theta \in [0, 2\pi[$

$\dot{r} = r(1-r)$

Polar coordinates on the plane.





$\vartheta(t) = \vartheta_0 + t$

Limit cycle

$z = 1$

Model of the mechanical clock

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x - 2\mu v \end{cases}$$

Eqs. for the harmonic oscillator + dissipative term.
$(\omega^2 = k/m)$

Take an initial point $(0, v_0)$, $v_0 > 0$

$$\ddot{x} = -\omega^2 x - 2\mu \dot{x}$$
$$\ddot{x} + 2\mu \dot{x} + \omega^2 x = 0$$

This eq. can be expl. solved and

$$x(t) = \frac{v_0}{\sigma} e^{-\mu t} \sin(\sigma t)$$

$x(0) = 0$, $\dot{x}(0) = v_0$

where $\sigma = \sqrt{\omega^2 - \mu^2}$

$$v(t) = v_0 \, e^{-\mu t} \left(-\frac{\mu}{\sigma} \sin(\sigma t) + \cos(\sigma t) \right)$$

$x(t)$ crosses the $v > 0$ axis periodically, with period $T = 2\pi/\sigma$

The corresponding velocities are :

v_0

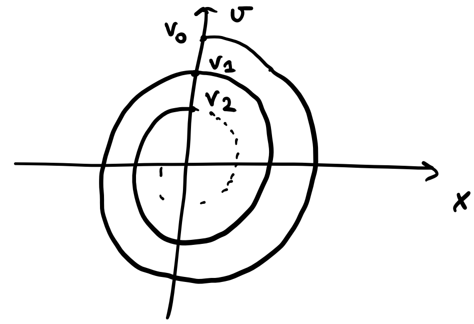
$v_1 = v_0 e^{-\frac{2\pi\mu}{\sigma}} = v_0 \, e^{-\mu T}$

$v_2 = v_1 \, e^{-\mu T}$

$v_3 = v_2 \, e^{-\mu T}$ ....

That is:

$$\begin{cases} v_{k+1} = a\, v_k \\ v_k = a^k v_0 \end{cases} \quad \text{with } a = e^{-\mu T} < 1$$



This model - without an external force - fated to stop ....

So - we add an external force as follows:

WHEN the point P of mass m passes through the $v>0$, it receives a positive impulse which increase the velocity of a fixed quantity:

In formulas:

v_0

$a v_0 \rightsquigarrow a v_0 + b = v_1$

$a v_1 \rightsquigarrow a v_1 + b = v_2$

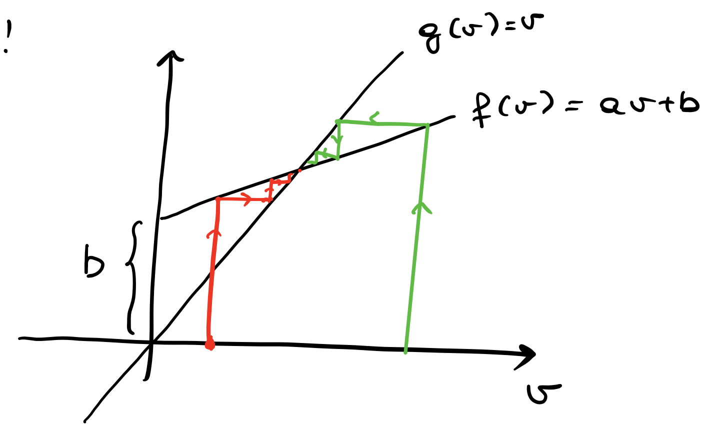
$a v_2 \rightsquigarrow a v_2 + b = v_3$

That is

$$v_{k+1} = a v_k + b \quad \forall\, k = \{0, 1, 2 \dots\}$$

where $a = e^{-\mu T} < 1$, $b > 0$ fixed.

Dynamics ?!



All solutions approach asymptotically the limit cycle (desired periodic solution of fixed amplitude).

$$f(v^*) = a v^* + b = v^* \iff v^* = \frac{b}{1-a} > 0$$

$1 - a > 0$ since $\underline{a < 1}$

— x — x — x —