Theorem:
\[ a_n \to l, \quad b_n \to l \]
1) \( l \in \mathbb{R} \quad l \in \mathbb{R} \]
\[ a_n + b_n \to l_1 + l_2 \]
2) \( a_n, b_n \to l_1, l_2 \)
3) \( \frac{a_n}{b_n} \to \frac{l_1}{l_2} \]
if \( l_2 \neq 0 \).
\((\text{if } l_2 > 0 \Rightarrow \exists N \text{ for } N \)
\[ b_n > 0 \]
\(\text{Property: If } c_n \to l \in \mathbb{R} \]
\[ \Rightarrow \{ c_n, n \in \mathbb{N} \} \text{ is bounded} \]
\( \text{Proof: } \exists \epsilon > 0 \quad \exists N \quad \forall n \geq N \quad \ell - 1 < C_n < \ell + 1 \)

\[
M = \max \left\{ \left| \ell \right| + 1, \left| \ell \right|, \ldots, \left| \ell \right| \right\}
\]

\(-M \leq C_n \leq M \quad \forall n \in \mathbb{N} \)
Theorem \( a_n \to l_1 \in \mathbb{R} \)
\( b_n \to l_2 \in \mathbb{R} \)
\[ \Rightarrow \quad a_n \cdot b_n \to l_1 \cdot l_2 \]

Proof

+ \( \varepsilon > 0 \)

\[ \exists N_1 : |a_n - l_1| < \frac{\varepsilon}{2A} \]

\[ \exists N_2 : |b_n - l_2| < \frac{\varepsilon}{2A} \]

\[ |l_1 \cdot l_2 - a_n b_n| = \]
\[ |l_1 l_2 - a_n l_2 + a_n l_2 - a_n b_n| \]
\[ \leq |l_1 l_2 - a_n l_2| + |a_n l_2 - a_n b_n| \]
\[ = |l_2| |l_1 - a_n| + |a_n||l_2 - b_n| \]
We know that \( \exists M > 0 \) such that \(|e_n| < M \) for all \( n \). Then

\[
|e_n| \leq |e_n - d_n| + M|e_2 - b_n|
\]

and

\[
A = (|e_2| + M)
\]

Thus,

\[
|e_n - d_n| + |e_2 - b_n| \leq A
\]

and

\[
\left(\frac{\varepsilon}{2A}\right) - \left(\frac{\varepsilon}{2A}\right) = \varepsilon
\]

Thus,

\[
|d_n - e_2| \leq \varepsilon
\]

and

\[
N := \max\{N_1, N_2\}
\]
What about infinite limits? How do they behave with operations?

Case 1: \( a_n \to l, b_n \to +\infty \)

\[ a_n + b_n \to +\infty \]

**Proof:**

\( \forall \varepsilon > 0 \) there exists \( N \) such that

\[ n > N \implies b_n > \frac{\varepsilon}{2} \]

\( a_n \to l \implies (a_n) \) is bounded

\[ |a_n| \leq K > 0 \text{ for } n > N \]

We want to show \( (a_n + b_n) \) is bounded.

\[ a_n + b_n \geq C \quad \forall n > N \]

Choose \( M = C + K \)

With some proof \((a_n)\) is bounded

\[ b_n \to +\infty \quad a_n + b_n \to +\infty \]
Let \( a_n \rightarrow l \in \mathbb{R} \) (or even less, i.e., \( a_n \) is bounded)
\[ b_n \rightarrow -\infty \]

\[ \Rightarrow a_n + b_n \rightarrow -\infty \]

\[
\begin{array}{c|c}
 a_n & a_n \rightarrow +\infty \\
 b_n & b_n \rightarrow +\infty \\
 a_n + b_n & a_n + b_n = +\infty
\end{array}
\]

\[
\begin{array}{c|c}
 a_n & a_n \rightarrow -\infty \\
 b_n & b_n \rightarrow -\infty \\
 a_n + b_n & a_n + b_n = -\infty
\end{array}
\]

\[ a_n = n + 27 \rightarrow +\infty \]
\[ b_n = -n \rightarrow -\infty \]
\[ a_n + b_n = 27 \rightarrow 27 \]
\[ a_n = n + (-1)^n \rightarrow +\infty \]

\[ b_n = -n \quad \frac{1}{4} \quad -\infty \]

\[ a_n = n^2 \quad a_n + b_n = n^2 - n = (-1)^n \]

\[ b_n = -n \quad \frac{1}{4} \quad -\infty \]

\[ a_n = -n^2 \quad a_n + b_n \rightarrow -\infty \]

\[ b_n = n \quad \frac{1}{4} \quad -\infty \]

\[ a_n + b_n \rightarrow -\infty \quad b_n \rightarrow -\infty \quad \text{indeterminate} \]

\[ a_n \rightarrow +\infty \quad b_n \rightarrow l \geq 0 \quad a_n \cdot b_n \rightarrow +\infty \]

\[ a_n \rightarrow +\infty \quad a_n b_n \quad \text{is indeterminate} \]
\[ a_n = n^4 \quad \rightarrow \quad +\infty \]
\[ b_n = \frac{1}{n^2} \]

\[ a_n \cdot b_n = \frac{1}{n} \rightarrow 0 \]
\[ a_n \cdot b_n = n \rightarrow +\infty \]

\[ d_n = n^4 \]
\[ b_n = \frac{1}{n^2} \]
\[ d_n \cdot b_n = 1 \]

\[ a_n \rightarrow l > 0 \]
\[ b_n \rightarrow 0 \]
\[ \frac{a_n}{b_n} \]

\[ b_n = (-1)^n \frac{n^2}{n^2} \]

\[ a_n \cdot b_n = a_n n^2 (-1)^n \]

This changes sign at every step.
\[ \lim_{n \to \infty} b_n = 0^+ \]

\[ \forall n \geq N \quad b_n > 0 \]

\[ \lim_{n \to \infty} b_n = 0 \]

Example: \[ \frac{1}{n} \quad \frac{1}{n^2 + 2n - 3} \quad \_ \]

\[ \text{Theorem:} \quad a_n \to \infty \]

\[ b_n \to 0^+ \]

\[ \Rightarrow \]

\[ \frac{a_n}{b_n} \to +\infty \]

\[ a_n \to +\infty \]

\[ b_n \to -\infty \]

\[ \text{and} \quad b_n \to +\infty \]

\[ a_n \cdot b_n \to -\infty \]

\[ \frac{a_n}{b_n} \to +\infty \]
\[ a_n \to +\infty \]
\[ b_n \to 0^+ \]

\[ \frac{a_n}{b_n} \to +\infty \]

\[ a_n \to +\infty \]
\[ b_n \to +\infty \]

\[
\frac{\sqrt{n}}{(n+3)^2} \to 0
\]

Numerator → +∞
Denominator → +∞

\[
\frac{n^2 + \sin n + \sqrt{n}}{n^2 + 6n + 9}
\]

\[
\frac{n^2 + \sin n + \sqrt{n}}{n^2 + \frac{6n}{n^2} + \frac{9}{n^2}}
\]

\[
\frac{1}{n^2 (1 + \frac{6}{n^2} + \frac{9}{n^2})}
\]

\[
1
\]
\[
\lim_{n \to \infty} \frac{n^3 \sin u + \sqrt{n}}{(n+3)^2} = \frac{n^2 \left( 1 + \frac{\sin u}{n^3} + \frac{1}{n^2} \right)}{10^2 \left( 1 - \frac{6}{n} + \frac{9}{n^2} \right)} \\
\xrightarrow{n \to \infty} 1
\]

**Def:** Suppose that \((a), (b)\) have infinite limit. \((a)\) is a higher order infinity than \(b\) if
\[
\lim_{n \to \infty} \frac{|a_n|}{|b_n|} = +\infty
\]
(a) and (b) are of the same order if \( \lim_{n \to \infty} \frac{a_n}{b_n} = C \neq 0 \)

(b) is asymptotic to (a) if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \)

If \( a > 1 \), \( x > 0 \), \( b > 1 \),

\( b^n \) is of higher order than \( n^a \) if \( n^a \) is of higher order than \( b^n \).

In the notation used in the notes,

\( \log_a n \ll n^x < b^n \)

\( \log_a n \ll n^x < b^n \)

Hence \( \lim_{n \to \infty} \frac{b^n}{n^a} = \infty \)

\( \lim_{n \to \infty} \frac{n^a}{\log_a n} = +\infty \)

\( \Rightarrow \lim_{n \to \infty} \frac{b^n}{\log_a n} = +\infty \)
Prove \[
\lim_{n \to \infty} \frac{b^n}{n^\alpha} = +\infty
\]

\[b = \frac{1 + h}{1 + h^\frac{1}{\alpha}} \geq 1 + nh \to +\infty\]

\[\frac{b^n}{n^\alpha} = (1 + h)^n \geq 1 + nh \to +\infty\]

\[\alpha > \frac{1}{2} \left( \frac{b^n}{n^\alpha} \right) = \left( \frac{b^{\frac{\alpha}{2\alpha}}}{n^{\frac{1}{2\alpha}}} \right)^{2\alpha} = \left( \frac{\left(1 + h^{\frac{1}{2\alpha}}\right)^n}{n^{\frac{1}{2\alpha}}} \right)^{2\alpha}\]

\[\sim b = b^{\frac{1}{2\alpha}} \quad \sim \frac{b}{n^{\frac{1}{2\alpha}}}\]

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We have used
\[c_n \to +\infty\]
\[c_n^\beta \to +\infty \quad \forall \beta > 0\]