• Define solutions and orbits for a differential equation.

• What is a vector field on an open set of $\mathbb{R}^n$? What does it mean that the vector field is complete?

• Define equilibria of a vector field. How can you dynamically characterize equilibria?

• How do you construct the phase portrait of $\dot{z} = X(z), z \in \mathbb{R}$?

• What is the “Allee effect” in population dynamics? Can you write a model including this effect in the logistic equation?

• Explain consequences of Existence and Uniqueness Theorem on orbits of differential equations.

• What can you say about a point $x^*$ in the phase space if there is a solution of a differential equations converging to $x^*$ for $t \to +\infty$ (or $-\infty$)?

• Be sure to draw all the phase portraits studied during the lectures.

• Enumerate the properties of the flow for a vector field.

• What is the linearization of a vector field around an equilibrium?

• What does it mean that a second order differential equation is equivalent to a system of first order differential equations?

• For a second order differential equation, write the general form of equilibria and explain the difference between equilibrium and equilibrium configuration.

• Let $k \in \mathbb{R}$. Draw the phase portrait of $\dot{z} = z^2 + k, z \in \mathbb{R}$, for $k < 0$, $k = 0$ and $k < 0$. In each case, compare the phase portrait near every equilibrium with the one of the linearization around the equilibrium.

• Determine equilibria of the system

\[
\begin{align*}
\dot{x} &= x - 2y + xy \\
\dot{y} &= 2x + y - y^2
\end{align*}
\]

and linearize the system around one of the equilibria.
• Determine equilibria of the system

\[
\begin{aligned}
\dot{x} &= 2(x - 1) + y + (x - 1)y \\
\dot{y} &= 1 - x + 2y + (x - 1)y^2
\end{aligned}
\]

and linearize the system around one of the equilibria.

• Is it possible that the differential equation \( \dot{z} = X(z) \) with \( z \in \mathbb{R} \) has periodic solutions? Justify the answer.

• Claim and proof of the theorem on the exponential divergence of solutions. Is this estimate “optimal”?

• What does it mean “sensitive dependence” on initial data? Introduce some models showing this behavior.

• How solutions of a differential equations depend on parameters?

• Define the matrix exponential.

• Be sure to deduce the phase portraits of linear systems on the plane, with diagonalizable matrix.

• Be sure to remind and interpret the bifurcation diagram of linear systems on the plane, with diagonalizable matrix.

• Draw the phase portrait of the linear system associated to the matrix

\[
\begin{pmatrix}
2 & -1 \\
1 & 2
\end{pmatrix}
\]

• Draw the phase portraits of the linear systems associated to

\[
\begin{pmatrix}
3 & -1 \\
2 & 2
\end{pmatrix}, \begin{pmatrix}
3 & -1 \\
-1 & 3
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
1 & -3
\end{pmatrix}, \begin{pmatrix}
3 & -1 \\
1 & 3
\end{pmatrix}.
\]

In the case of nodes and saddles, determine also the stable and unstable eigenspaces.

• Define elliptic and hyperbolic equilibria. What can we say for the phase portrait, locally around these equilibria, for a non-linear differential equation?

• Define first integrals for a differential equation.

• Define the Lie derivative of a function \( f \) along a vector field \( X \).
• What does it mean that $c \in \mathbb{R}$ is a regular value for a function $f$? Explain consequences of this property on the corresponding level set.

• Define (topological) stability, unstability and asymptotic stability for an equilibrium.

• Claim and prove Lyapunov Theorems on simple and asymptotic stability.

• How do you draw the phase portrait for a 1-dim Newton equation $m\ddot{x} = -V'(x)$? Where energy levels are regular manifolds? How these levels intersect the $x$-axis? Explain in details.

• Be sure to deduce all the phase portraits of 1-dim Newton equations $m\ddot{x} = -V'(x)$ studied during lectures.

• Let $f : \mathbb{R} \to \mathbb{R}$. Which is the difference between the phase portrait of 
\[ \dot{z} = f(z), \quad z \in \mathbb{R} \]
and the phase portrait of 
\[ \ddot{x} = f(x), \quad x \in \mathbb{R}? \]

• Let consider the system
\[
\begin{aligned}
\dot{x} &= x(1 - x - y) \\
\dot{y} &= y(2 - x - y)
\end{aligned}
\]

1. Determine equilibria, linearize the system around each equilibrium and draw the phase portraits of the linearized systems.

2. In such a case, is it possible to deduce the phase portrait of the original system?

• Describe qualitatively the motion of a point of mass $m$ subjected to a 1-dim conservative force with potential:
\[ V(x) = (2x^2 - x)e^{-x/2} \]
or
\[ V(x) = x \sin x. \]

• Let consider a point of mass $m = 1$ subjected to a 1-dim conservative force with potential $V(x) = -(x^2 - 1)^2$. For the energy level $E = 0$, calculate the time needed to go from $x_0 = 0$ to $x_1 = 1$, assuming that the initial velocity is positive.
• What does it mean that a dynamical system has a bifurcation?

• Other interesting exercises, you can have a curious look to:
  From 3.1.1 to 3.1.5 (pages 79-80), from 3.2.1 to 3.2.4 (page 80), from 3.4.1 to 3.4.10 (pages 82-83), from 5.1.1 to 5.1.9 (pages 140-141), from 5.2.1 to 5.2.10 (pages 142-143), Example 6.5.2 in “Nonlinear Dynamics and Chaos”, S.H. Strogatz.