Lesson 10 - 19/11/2022

1) Bifurcation diagram for $\dot{x} = x^2 + \mu x + 1$, $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

2) $\ddot{x} = -\omega^2 \sin x - k\dot{x}$ $(\omega > 0$ and $k \geq 0)$. Equilibria & their stability.

3) $\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x - y^3 \end{cases}$

Stability of $(0,0)$ by one of the next candidate Lyapunov functions:
$W_1(x,y) = x^2 - y^2$, $W_2(x,y) = x^2 + y^2$, $W_3(x,y) = x^3 + y^3$.

4) $m\ddot{x} = f(x)$ is the "prototype" of a physical system with conservative forces and 1-dim ideal constraint.

↙ General facts

↘ Phase portraits of the harmonic oscillator and the harmonic repeller.

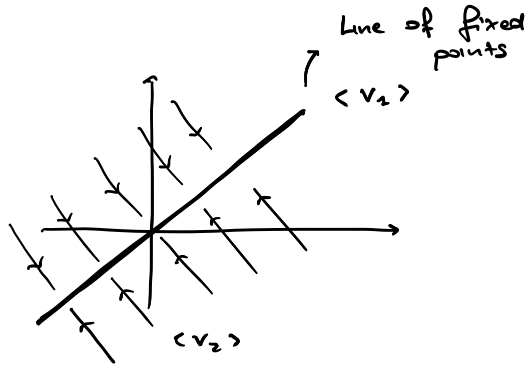
Remark 1 (Question 1)

$$a\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$ (Love affairs, case $b = -a$, $a > 0$)

Eigenvalues: $0, -2$

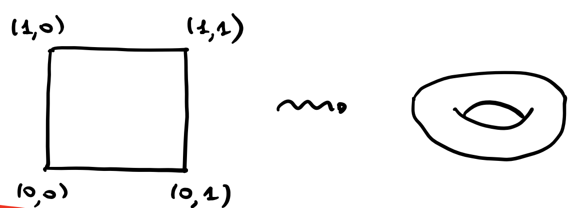
$0 \to v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $-2 \to v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ The phase portrait is the following



Remark 2 (Question 2)

Let $\mathbb{T}^2 :=$ flat torus
$\pi : \mathbb{R}^2 \to \mathbb{T}^2$, $z \mapsto [z]$



$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$ Arnold's cat

$\psi_A([z]) = [Az]$ and its iterations.

$\lambda_1 = 1 + \sqrt{2}$ with $v_1 = (\sqrt{2}, 1)$ (expansion)

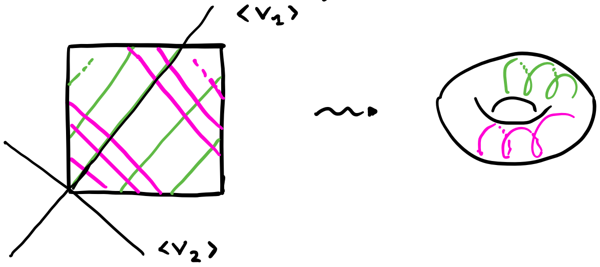
$\lambda_2 = 1 - \sqrt{2}$ with $v_2 = (\sqrt{2}, -1)$ (contraction)

$(0,0)$ is a saddle (hyperbolic equilibrium)

It holds:

$W^s(0,0) = \{[t v_2],\ t\in\mathbb{R}\}$

$W^u(0,0) = \{[t v_1],\ t\in\mathbb{R}\}$



- 2 (irrational) helixes on the torus, both dense on $\mathbb{T}^2$.
- Also $W^s(0,0)\cap W^u(0,0)$ is dense on $\mathbb{T}^2$.

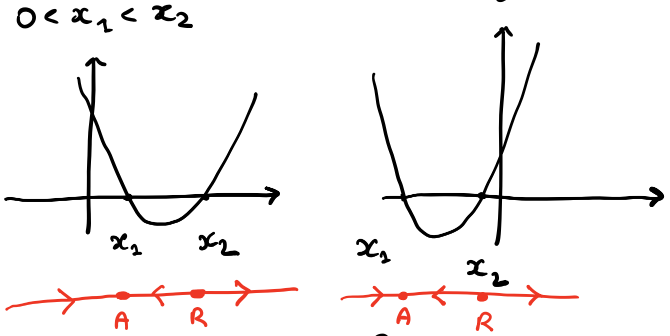
EX 1 Bifurcation diagram for $\dot{x} = x^2+\mu x+1$, $x\in\mathbb{R}$, $\mu\in\mathbb{R}$

SOL $x^2+\mu x+1=0 \iff x_{1,2} = \dfrac{-\mu\pm\sqrt{\mu^2-4}}{2}$

$\boxed{1}$ $\mu\in\underbrace{(-\infty,-2)}\cup(2,+\infty)$: 2 real solutions (2 distinct equilibria)

$(-\infty,-2) \downarrow$ $0<x_1<x_2$

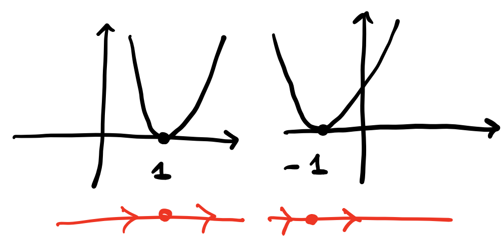
$(2,+\infty) \downarrow$ $x_1<x_2<0$



$\boxed{2}$ $\mu=-2$ OR $\mu=+2$

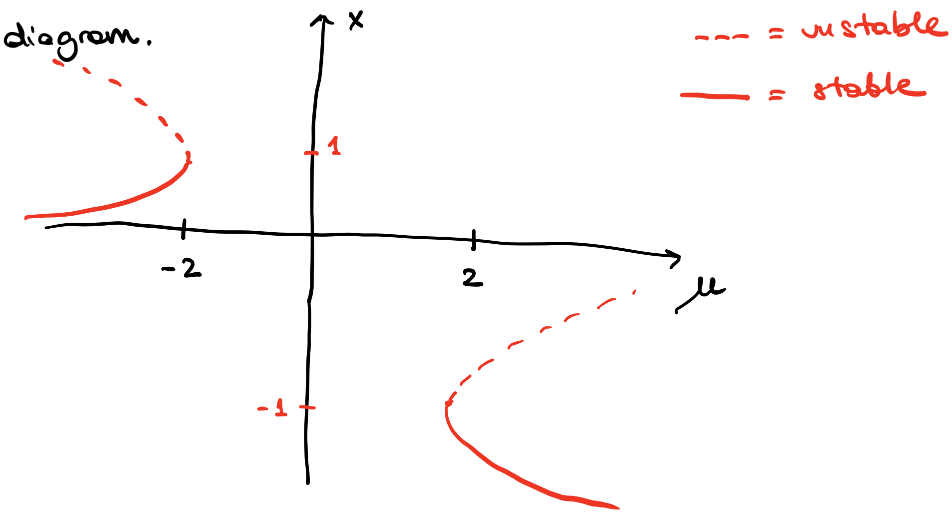
$\mu=-2 \downarrow$ $x_1=x_2=1$

$\mu=+2 \downarrow$ $x_1=x_2=-1$



$\boxed{3}$ $-2<\mu<+2$, NO EQUILIBRIA.

Bif. diagram.



EX 2 $\ddot{x} = -\omega^2 \sin x - k\dot{x}$ $\quad \begin{pmatrix} k \geq 0 \\ \omega > 0 \end{pmatrix}$ Pendulum with friction.

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 \sin x - kv \end{cases}$$

Equilibria in $x \in [0, 2\pi[\; \to (0,0)$ AND $(\pi, 0)$

$$Jx(x,v) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -k \end{pmatrix}$$

$$Jx(0,0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -k \end{pmatrix} \to \begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x - kv \end{cases}$$

Linear. around $(0,0)$

Eigenvalues for $Jx(0,0)$ are $\lambda_{1,2} = \dfrac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$

$\boxed{k=0} \to$ No friction No infos from the linearization.

But in such a case we can use an approp. Lyap. function.

$E(x,v)$ = total energy (which is conserved).

$$\ddot{x} = \underbrace{-\omega^2 \sin x}_{V'(x)} = -V'(x) \Rightarrow V'(x) = \omega^2 \sin x$$

$$\Rightarrow V(x) = -\omega^2 \cos x$$

$$E(x,v) = \frac{1}{2} v^2 - \omega^2 \cos x \underbrace{-V(0)}_{+\omega^2}$$

$E(0,0) = 0$ and > 0 outside $(0,0)$.

$$L_X E(x,v) = \omega^2 \operatorname{sen} x \, (\dot{x}) + v\dot{v} = \omega^2 \operatorname{sen} x \, (v) + v(-\omega^2 \operatorname{sen} x) \equiv 0$$

$\Rightarrow (0,0)$ is top. stable.

$\boxed{K>0}$ $\quad \lambda_{1,2} = \dfrac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$

- if $k^2 - 4\omega^2 > 0$: 2 REAL < 0
- if $k^2 - 4\omega^2 = 0$: 1 REAL < 0
- if $k^2 - 4\omega^2 < 0$: 2 COMPLEX EIGENV. WITH < 0 REAL PART

IN EVERY CASE WE CAN CONCLUDE THE ASYMP. STABILITY OF $(0,0)$.

Eigenvalues of $JX(\pi,0)$

$$JX(\pi,0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & -k \end{pmatrix}$$

with eigenvalues:

$$\lambda_{1,2} = \frac{-k \pm \sqrt{k^2 + 4\omega^2}}{2} \rightarrow \text{SADDLE} \Downarrow \text{UNSTABLE}$$

(in every case, also for $K=0$)

ALWAYS UNSTABLE

EX 3 The unique function which is strictly positive definite around the origin is $W_2(x,y)=x^2+y^2$.

$$L_X W_2(x,y) = (2x, 2y)\cdot(-y-x^3,\ x-y^3) =$$

$$= -2xy - 2x^4 + 2xy - 2y^4 = -2(x^4+y^4) < 0$$

on $\mathbb{R}^2$ and $=$ in $(0,0)$.

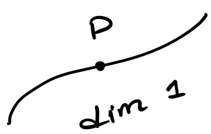
$$\Downarrow$$

$(0,0)$ is asymptotically stable

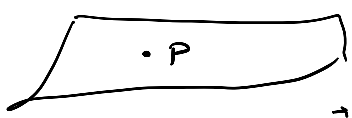
——x——x——x——

Phase-portraits of 1-dim conservative systems

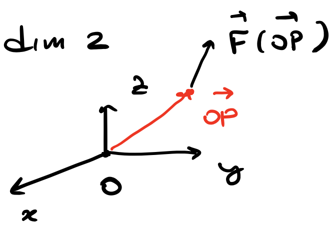
Physical problem: P point of mass m in $\mathbb{R}^3$ subj. to a positional force field. P is also constrained.



dim 1



dim 2



$Oxyz$, $\vec{OP}$ = position of the point.

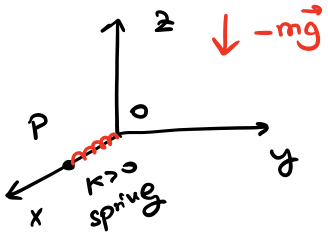
Moreover, the positional force field is given by a function:

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{OP} \longmapsto \vec F(\vec{OP})$$

The dynamics is given by Newton's law:

$$m\,\ddot{\vec{OP}} = \underbrace{\vec F(\vec{OP})}_{\text{positional forces}} + \underbrace{\vec\Phi}_{\text{constraints.}}$$

EXAMPLE



Constraint : x-axis $\rightarrow$

Constraint is IDEAL : $\vec\phi = (\phi_1, \phi_2, \phi_3)$

s.t. $\vec\phi \cdot \vec e_1 \equiv 0 \iff \phi_1 = 0$

$$m\ddot{\vec{OP}} = \vec F(\vec{OP}) + \vec\phi$$

projecting on the coord. axes: $\vec{OP} = (x, 0, 0)$

$$\begin{cases} m\ddot x = -Kx + \overbrace{\phi_1}^{=0} \rightarrow m\ddot x = -Kx \\ 0 = \phi_2 \\ 0 = -mg + \phi_3 \rightarrow \boxed{\phi_3 = mg} \end{cases}$$

$m\ddot x = -Kx$ of type : $\boxed{m\ddot x = f(x)} = \vec F(\vec{OP}) \cdot \vec e_1$

- The force field $\vec{OP} \mapsto \vec F(\vec{OP})$ is called conservative if $\exists$ function $V : \mathbb{R}^3 \to \mathbb{R}$, $\vec{OP} \longmapsto V(\vec{OP})$ s.t.

$$\vec F(\vec{OP}) = -\nabla V(\vec{OP}).$$

- In the sequel, we will study the case $n = 1$ with f continuous.

$$V(x) = -\int_a^x f(t)\,dt \Rightarrow -V'(x) = f(x) \quad \forall x \in \mathbb{R}.$$

$$\downarrow$$

$$\boxed{m\ddot x = -V'(x)}$$

$\downarrow$ V = potential energy

REMARKS

1) Configuration space : $\mathbb{R} = \mathbb{R}_x$
Phase-space : $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_v$

2) The potential energy is def. up to an additive constant.

3) $$\begin{cases} \dot x = v \\ \dot v = -V'(x)/m \end{cases} \qquad (*)$$

4) The dyn. system $(*)$ admits the total energy as first integral:

$$E(x,v) = \frac{1}{2} m v^2 + V(x)$$

PROOF

$$L_X E(x,v) = \nabla E(x,v) \cdot X(x,v) = \left(V'(x), mv\right) \cdot \left(v, -\frac{V'(x)}{m}\right) = V'(x) v - v V'(x) \equiv 0. \qquad \square$$

We can use the total energy to draw the phase-portraits !!

$$N_c = \{ (x,v) \in \mathbb{R}^2 : E(x,v) = c \}$$

$\downarrow$

level set for E

Recall that the total energy is conserved! So, if

$$t \mapsto (x_t, v_t) = (x_t, \dot{x}_t)$$

is a solution of $(*)$ passing throught $(x_0, v_0) = (x_0, \dot{x}_0)$

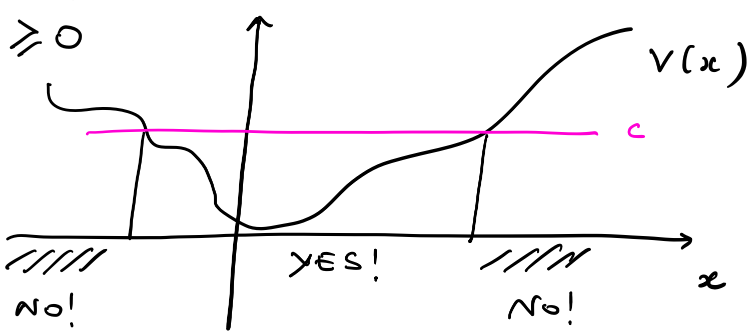
then

$$E(x_0, \dot{x}_0) = \frac{1}{2} m \dot{x}_0^2 + V(x_0) = c$$

$$\Downarrow$$

$$E(x_t, \dot{x}_t) = \boxed{\frac{1}{2} m \dot{x}_t^2 + V(\dot{x}_t) = c} \qquad \forall t \in \mathbb{R}$$

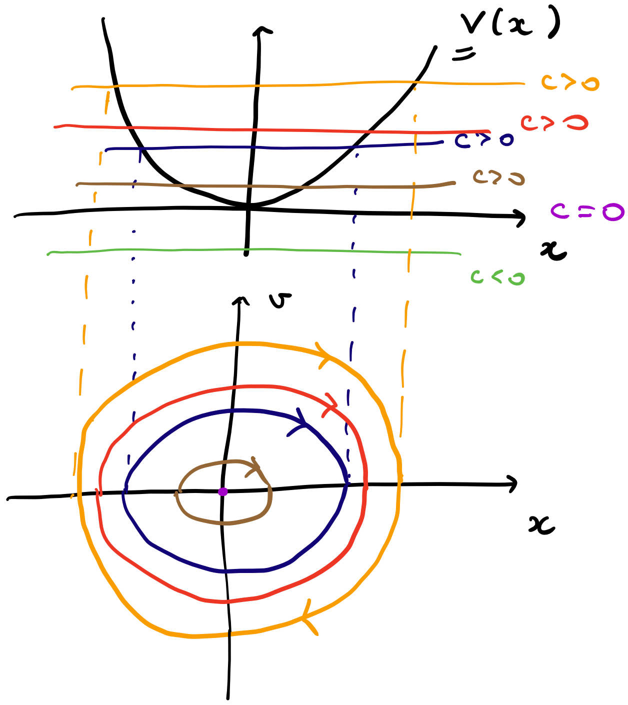
$$\Rightarrow \underbrace{\frac{1}{2} m \dot{x}_t^2}_{\geq 0} = c - V(x_t) \geq 0 \iff V(x_t) \leq c \quad \forall t \in \mathbb{R}.$$



1 HARMONIC OSCILLATOR / REPELLER

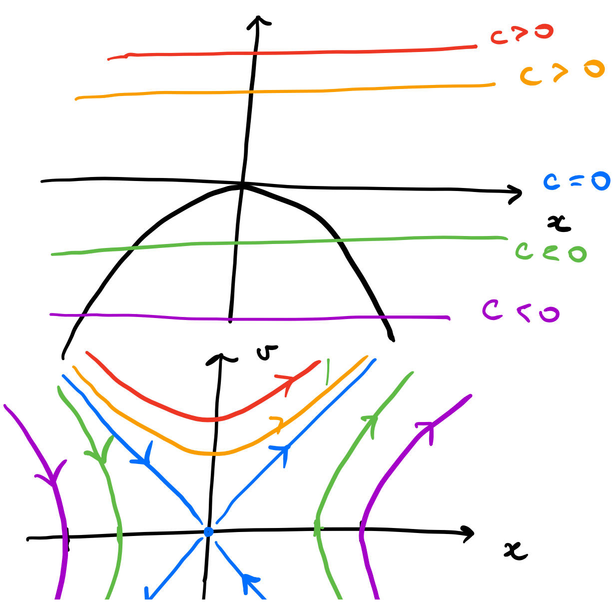
- $m\ddot{x} = -kx = -V'(x) \to V(x) = \frac{1}{2}kx^2 \qquad k>0$

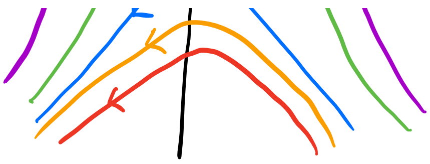
$$E(x,v) = \frac{1}{2}mv^2 + \underbrace{\frac{1}{2}kx^2}_{=V(x)}$$



- $m\ddot{x} = kx \Rightarrow V(x) = -\frac{1}{2}kx^2 \qquad k>0$

$$E(x,v) = \frac{1}{2}mv^2 \underbrace{-\frac{1}{2}kx^2}_{V(x)}$$





Remarks

- $\frac{1}{2}m\dot{x}^2 + V(x) = C \Rightarrow \dot{x}^2 = \frac{2(C - V(x))}{m}$

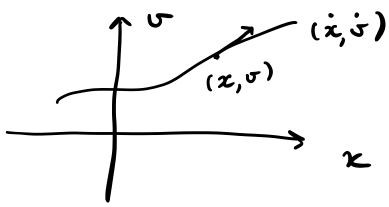
$$\Rightarrow \dot{x} = \pm\sqrt{\frac{2(C-V(x))}{m}}$$

This means that the phase-portraits are symmetric with respect to the x-axis.

- Let $t \mapsto (x(t), v(t))$ be a solution of $(*)$.

  then $\dot{x}(t) = v(t) \quad \forall t \in \mathbb{R}$.

  what does it mean that an energy level has vertical tangent ?!



Vertical tangent means that the vector $(\dot{x},\dot{v})$ is of type $(0, \dot{v}) \Leftrightarrow \dot{x} = 0 \Leftrightarrow v = 0$.

This means that orbits of $(*)$ have vertical tangent only when they cross the x-axis ("inversion points")

—x—x—x—

- $m\ddot{x} = -mg \rightarrow V(x) = mgx$
- $m\ddot{r} = -\frac{GMm}{r^2} \rightarrow V(z) = -\frac{GMm}{z}$