Lesson 9 - 17/10/2022

- Lyapunov theorem for asymptotic stability, with proof.
- Harmonic oscillator with friction: asymptotic stability of \((0,0)\) by an appropriate Lyapunov function and by the first Lyapunov method.
- Rabbits/sheep - Principle of competitive exclusion.
- \(\dot{x} = x^3 (x-2)(x+3)\)
- \(\dot{x} = x^2 + \mu x + 1\), \(x \in \mathbb{R}\) and \(\mu \in \mathbb{R}\).

**Lyapunov Theorem for Asymptotic Stability**

\(\bar{x} \in \mathbb{R}^n\), equilibrium for a v.f. \(\dot{z} = X(z)\).

Let suppose that \(\exists \ W(z) \in \mathbb{R}\) on \(A \supseteq \bar{x}\) (open neighborhood of \(\bar{x}\)) such that:

1. \(W(x) = 0\) and \(W(x) > 0\ \forall \ x \in A \setminus \{\bar{x}\}\).
2. \( \begin{cases} L_x W(x) < 0 \ \forall \ x \in A \setminus \{\bar{x}\} \\ L_x W(\bar{x}) = 0 \end{cases} \)

Then \(\bar{x} \in \mathbb{R}^n\) is asymptotically stable.

**Proof**

Condition (2) \(\Rightarrow\) (2) on topological stability of \(\bar{x}\).

So \(\bar{x}\) is top. stable. This means that fixed a neigh.

If \(U \supseteq \bar{x} (U \subseteq A)\) then \(\exists \ V \supseteq \bar{x} (V \subseteq A)\) such that

\(X_0 \in V \Rightarrow \phi^t(x_0) \in U \ \forall \ t \geq 0\).

But in order to prove top asympt. stability:

\(\lim_{t \to +\infty} \phi^t(x_0) = \bar{x}\)

\(<\Rightarrow \ W(\lim_{t \to +\infty} \phi^t(x_0)) = W(\bar{x}) \leq 0 \ \downarrow \ \text{since \ W \ is \ continuous} \)

\(<\Rightarrow \ \lim_{t \to +\infty} W(\phi^t(x_0)) = W(\bar{x}) = 0 \ \downarrow \ \text{By assump.} \)

But control, suppose that the previous limit doesn't hold.

Since \(W(x) > 0\ \forall \ x \in A \setminus \{\bar{x}\}\), this means that
\[
\lim_{t \to +\infty} w(\psi_t(x_0)) = \lambda > 0
\]

Now we write this equality by using the def. of limit of a real function for \( t \to +\infty \),

\[
\text{EQUALS TO:}
\]

\[
\forall \varepsilon > 0 \exists T_\varepsilon > 0 \text{ such that }
\]

\[
\forall t \geq T_\varepsilon \implies |w(\psi_t(x_0)) - \lambda| < \varepsilon
\]

that is

\[
\forall t \geq T_\varepsilon \implies \lambda - \varepsilon \leq w(\psi_t(x_0)) \leq \lambda + \varepsilon
\]

But, recall that \( t \mapsto w(\psi_t(x_0)) \) is strictly decreasing.

Then

\[
\forall t \geq T_\varepsilon \implies \lambda \leq w(\psi_t(x_0)) \leq \lambda + \varepsilon
\]

\( \lambda \) is the "\( \text{lim}^\dagger \)" value, so \( \varepsilon \mapsto w(\psi_t(x_0)) \)

cannot assume smaller value than \( \lambda \).

That is \( (\text{fixed } \varepsilon > 0) \}

\[
\psi_t(x_0) \in B := \left\{ x \in \mathbb{R}^n : \lambda \leq w(z) \leq \lambda + \varepsilon \right\}
\]

\( \forall t \geq T_\varepsilon \)

\( B \) is a compact set.

\( \bar{x} \notin B ! \)

As a consequenc:e

\[
\max_{x \in B} L_x w(x) \leq -\alpha < 0
\]

\( \forall x \in B \text{ } \forall \text{ by cond. 2} ! \)

\[
\implies \max_{x \in B} L_x w(x) \leq -\alpha < 0 \forall x \in B
\]

\[
\frac{d}{dt} w(\psi_t(x_0))
\]

\[
\implies w(\psi_{T_\varepsilon}^t(x_0)) - w(\psi_{T_\varepsilon}(x_0)) \leq -\alpha \varepsilon \forall \varepsilon > 0
\]

\[
\implies w(\psi_{T_\varepsilon}^t(x_0)) \leq w(\psi_{T_\varepsilon}(x_0)) - \alpha \varepsilon \forall \varepsilon > 0
\]

\[
\leq \lambda + \varepsilon
\]
But observe now that
\[ x + \lambda - \alpha \tau < \lambda \iff \tau > E/\alpha \quad \forall \tau > 0 \]
Since
\[ y(x, t) \in E \quad \forall \tau \geq T_e. \]
This is the desired contradiction, so
\[ \lim_{t \to +\infty} W(y^t(x_0)) = \lambda > 0 \]

THE LIMIT MUST BE 0. So \( \lambda \) is also a basin of attraction, so we have the asymptotic stability. \( \square \)

\((0, 0)\) is an asymptotically stable equilibrium for the harmonic oscillator with friction

\[ \begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x - 2\mu v
\end{align*} \]

By using (last week) \( E(x, v) = \frac{1}{2} v^2 + \frac{1}{2} \omega^2 x^2 \)
we obtained
\[ L \cdot W(x) = -2\mu v^2 \leq 0 \quad \text{on every neigh of } (0, 0). \]

\( L \cdot W(x) \) proves only simple stability.

\[ E(x, v) = \frac{1}{2} v^2 + \frac{1}{2} \omega^2 x^2 + \frac{1}{2} (v + 2\mu x)^2 + \frac{1}{2} \omega^2 x^2 \]

\( E(x, v) \)

\[ F(x, v) > 0 \quad \text{on a neigh of } (0, 0). \]

\[ F(0, 0) = 0. \]

\[ L \cdot F(x, v) = [\omega^2 x + (v + 2\mu x) 2\mu + \omega^2 x] \dot{x} + \\
+ [v + (v + 2\mu x)] \dot{v} = \]
... = \left( -2 \mu \right) \left( v^2 + w^2 x^2 \right) < 0 \\

But = 0 \text{ IFF } \left( x, v \right) = (0, 0), \text{ if sub. }

the expr. \text{ for } \begin{cases} x = v \\
v = -w^2 x - 2 \mu x \end{cases}

=0 \text{ (0,0) is ASYMPTOTICALLY STABLE for the harm. oscill. with friction.}

OTHER WAY: First Lyapunov method !!

\text{JX} \left( x_1, y \right) = \begin{pmatrix} 0 & 1 \\ -w^2 & -2\mu \end{pmatrix}

\text{det} \begin{pmatrix} -\lambda & 1 \\ -w^2 & -2\mu - \lambda \end{pmatrix} = 0

\Rightarrow 2\mu \lambda + \lambda^2 + w^2 = 0

\lambda_{1,2} = \frac{-2\mu \pm \sqrt{4\mu^2 - 4w^2}}{2} = -\mu \pm \sqrt{\mu^2 - w^2}

\begin{align*}
\bullet \mu^2 - w^2 > 0 & \text{ in such a case } \sqrt{\mu^2 - w^2} < \mu \\
\Rightarrow 2 \text{ real } < 0 \text{ eigenvalues!} \\
\bullet \mu^2 = w^2 & \Rightarrow \lambda_{1,2} = -\mu < 0 \\
\bullet \mu^2 - w^2 < 0 & \Rightarrow 2 \text{ complex conjugate eigenvalues } \\
& \text{ with } < 0 \text{ real part.}
\end{align*}

ASYMPTOTIC STABILITY!

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When rabbits and sheep encounter each other, trouble starts. Since the two populations are in competition for food, we assume that the conflicts occur at a rate proportional to the size of each population.
Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for rabbits.

\[
\begin{align*}
\dot{x} &= x(3-x-2y) \\
\dot{y} &= y(2-y-x)
\end{align*}
\]

\[x(t) = \text{pop. of rabbits}, \quad y(t) = \text{pop. of sheep}, \quad x, y \geq 0\]

Study the dynamics.

**Solution**

**First step. Equilibria**

\[
\begin{align*}
x(3-x-2y) &= 0 \\
y(2-y-x) &= 0
\end{align*}
\]

First case

\[
\begin{align*}
x &= 0 \\
y(2-y) &= 0
\end{align*}
\]

\((-0,0) \text{ or } (0,2)

Other case

\[
\begin{align*}
3-x-2y &= 0 \\
y(2-y-3+2y) &= 0
\end{align*}
\]

\((3,0) \text{ or } (1,1)

Four equilibria for the non-linear v.f.

\((0,0), (0,2), (3,0), (1,1)

\[
A = \begin{pmatrix}
3-x-2y & -2x \\
-2y & 2-x-y-y
\end{pmatrix} = \begin{pmatrix}
3-2x-2y & -2x \\
-2y & 2-x-2y
\end{pmatrix}
\]
\[ A(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \] is an unstable node

\[ \lambda_1 = 2 \Rightarrow v_1 = (0, 1) \]
\[ \lambda_2 = 3 \Rightarrow v_2 = (1, 0) \]

\[ A(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \]

with eigenvalues:
\[ \lambda_1 = -1 \text{, } v_1 = (1, -2) \]
\[ \lambda_2 = -2 \text{, } v_2 = (0, 1) \]

\((0, 2)\) is a stable node!

\[ A(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \]

\[ \lambda_1 = -3 \text{ and } v_1 = (1, 0) \]
\[ \lambda_2 = -1 \text{ and } v_2 = (3, 1) \]

\((3, 0)\) is a stable node.

\[ A(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \]

\[ \det = 1 - 2 < 0 \Rightarrow \text{saddle point!} \quad (\lambda_1, 2 = -1 \pm \sqrt{2}) \]
Essentially in every case, one species drives the other to extinction! → Biological interpretation.

**Principle of Competitive Exclusion:**
2 species competing for the same limited resource cannot coexist!!

**EX 1** \( \dot{x} = x^3(x-1)(x+3) \)

1) Find equilibria
2) Linearize the v.f. around the eq. \( x_0 > 0 \) and discuss its stability.
3) Can we use \( x^2 \) as Lyapunov function to prove the stability of \( x_0 = 0 \)?

**Sol** \( x^3(x-1)(x+3) = 0 \)

\[ \begin{align*}
  x & = 0 \\
  x & = 1 \\
  x & = -3
\end{align*} \]

\( x'(x) = 3x^2(x-1)(x+3) + x^3(x+3) + x^3(x-1) \)

\( = 0 \)

\( x'(1) = 4 = 0 \)

\( \dot{x} = 4(x-1) \)

\( \dot{x'} > 0 \)

\( = 0 \) is unstable.

\( \dot{x'(0)} = 0 = 0 \) no w.p.s on 0 by the first derivative.
$W(x) = x^2$ is a good candidate Lyapunov function for $x_0 = 0$.

$L_x W(x) = 2x \frac{d}{dx} = 2x^4 (x-1)(x+3) < 0$ in $\mathbb{R}$.

$\geq 0$ parabola. $[-3, 1 \cup \{0\}]$.

$= 0$ 0 IS ASYMPT. STABLE!

EX2 Bifurcation diagram for $\dot{x} = x^2 + \mu x + 1$,
$x \in \mathbb{R}$, $\mu \in \mathbb{R}$.

Sol $\mu(x) = x^2 + \mu x + 1 = 0$.

$x_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2} \iff \mu^2 - 4 > 0$

$\iff \mu < -2 \text{ or } \mu > 2$.

In particular,
1. If $\mu \in (-\infty, -2) \cup (2, +\infty)$: 2 real, distinct equilibria.
2. If $\mu = \pm 2$, we have $x_1 = x_2$
3. If $\mu \in (-2, 2)$, no equilibrium.

\[x\]