Lesson 7 - 12/10/2022

Thank you to the student who corrected me on the direction of rotation! I made a trivial mistake!

Recall the formula (in the case of 2 complex eigenvalues $\lambda \pm i\beta$):

$$\phi_t(z_0) = 2 \phi e^{t \lambda} \left[ \cos(\lambda t + i\beta) u - \sin(\lambda t + i\beta) w \right]$$

- Suppose $u \rightarrow \tilde{w}$, then the rotation direction is clockwise.

  [If $\phi_0(z_0) \in \langle u \rangle$ then, for $t > 0$, $\cos(\lambda t + i\beta)$ has sign + but $\sin(\lambda t + i\beta)$ has sign -!!]

- With the same argument, when $u \rightarrow \tilde{w}$ then the rotation direction is counterclockwise.

Today: Exercises

Date for the first written exam:

11 Friday / 11 (10:30)

(in the morning)

IN TORRE ARCHIMEDES in Math Dep.

For ex. -0. Repeat the ones done in class.

- Try to solve some ex. from STROGATZ book. (Link)
EX 1

Draw the bifurcation diagram for:

\[ x = 2 - x - e^{-x}, \quad x \in \mathbb{R} \]

\[ x \in \mathbb{R} \]

Sol

\[ X(x) = 2 - x - e^{-x} = 0 \]

\[ X_e(x) \]

\[ \iff 2 - x = e^{-x} \implies \text{We adopt a geometric approach to det.} \]

\[ (\text{not explicitly) equilibr.} \]

\[ X(x) = 2 - x - e^{-x} > 0 \iff 2 - x > e^{-x} \]

We need to det. the bif, case = 0 we need to impose that the graphs of \( 2 - x \) and \( e^{-x} \) intersect tangentially.

\[
\begin{align*}
\begin{cases}
  e^{-x} = 2 - x \\
  -e^{-x} = -1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\iff \begin{cases}
  2 = 1 \\
  x = 0
\end{cases}
\end{align*}
\]
Then, the following cases occur:

- $z \in ]-\infty, 1[ : \text{no equilibrium}$

- $z = 1 : \exists! \text{equilibrium } x = 0$
  
  and $x(z) < 0$, $x \neq 0$.

- $z > 1 : \exists 2 \text{equilibria}$.

**Bifurcation diagram.**

**Ex. 2**

\[ x = -x + \beta \tanh x \]

where $x \in \mathbb{R}$, $\beta > 0$.

**Sol.**

\[ X(z) = -x + \beta \tanh x \]

\[ x(z) \]

$X(z) > 0 \iff x < \beta \tanh x$

we use the same argument of the previous ex.
\[
\begin{cases}
  x = \beta \tanh x \\
  1 = \beta \frac{1}{\cosh^2 x}
\end{cases}
\quad \Rightarrow \quad
\begin{cases}
  x = 0 \\
  \beta = 1
\end{cases}
\]

[or by recalling that \( \tanh x = x + o(x) \) ...]

So the next three cases occur:

\( 0 < \beta < 1 \)

1 equilibrium, \( x = 0 \), \( x(x) = -x + \beta \tanh x < 0 \)

for \( x > 0 \)

\( \beta > 1 \)

3 equilibria, \( 0, x_1 < 0, x_2 > 0 \)

\( \Rightarrow \) repeller
Ex 3: Love affairs! (I)

\[
\begin{aligned}
\dot{R} &= aJ \\
\dot{J} &= -bR
\end{aligned}
\]
\[a, b > 0\]

\[A = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \quad \Rightarrow \quad \lambda_{1,2} = 1 \pm \sqrt{ab}\]

\[\Rightarrow (0,0) \quad \text{(the unique equilibrium is a center)}\]

A neverending cycle of love and hate!

Ex 4: Love affairs! (II)
\[
\begin{align*}
\dot{R} &= aR + bJ \\
\dot{J} &= bR + aJ
\end{align*}
\]
where \( a < 0 \) \hspace{1cm} b > 0

This model means that - in such a case - Romeo and Juliet are "cautious" loves!

In particular:
- \( a < 0 \) measures the rate of cautiousness.
- \( b > 0 \) measures the rate of responsiveness.

\[
A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}
\]

Eigenvalues are given by \( \lambda^2 - 2a\lambda + a^2 - b^2 = 0 \)

(\( \lambda \)) \( \lambda_1, 2 = a \pm \sqrt{b^2} = a \pm b \)

\( \lambda_1 = a + b \) has eigenvector \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

\( \lambda_2 = a - b \) has eigenvector \( \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

**First Case**

\( a - b < 0 \) always.

\( a + b > 0 \) \( \iff \) \( b > -a \) =D we obtain a saddle!
SECOND CASE

\( a - b < 0 \) always

\( a + b < 0 \) \( \Rightarrow \) \( b = -a \)  

\( (\mathbf{0}, 0) \) is a stable node

\( \Rightarrow \) The relation always finishes in mutual indifference!

[We don't analyse the case \( b = -a \)

\( \rightarrow A = a \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \), not diag.]
THE EFFECT OF SMALL NONLINEAR TERMS - part I -

Find equilibrium for the system
\[
\begin{align*}
  \dot{x} &= -x + x^3 \\
  \dot{y} &= -2y
\end{align*}
\]
and use linearization to classify them.
Then check that the conclusions on
the linear systems to draw the phase portrait for the full non-linear case.

\[
\begin{align*}
  -x + x^3 &= x(-1 + x^2) = 0 \\
  y &= 0
\end{align*}
\]
\[
x = 0, \quad x = \pm 1
\]

\[
P = (0, 0), \quad (1, 0), \quad (-1, 0)
\]

\[
J_X(x, y) = \begin{pmatrix}
-1 + 3x^2 & 0 \\
0 & -2
\end{pmatrix}
\]

\[
J_X(0, 0) = \begin{pmatrix}
-1 & 0 \\
0 & -2
\end{pmatrix} \rightarrow (0, 0) \text{ is a stable node}
\]

\[
J_X(\pm 1, 0) = \begin{pmatrix}
2 & 0 \\
0 & -2
\end{pmatrix} \rightarrow (\pm 1, 0) \text{ are both saddle points}
\]

But, in such a case, we can also
check explicitly the behavior of solutions for the original nonlinear system, since eqs. are uncoupled.

\[ y = -2y \rightarrow \text{In the } y\text{-direction, all trajectories decay exp. to } 0. \]

\[ x = x (-1 + x^2) \]

For the system on the plane, the lines \( x = 0 \) and \( x = \pm 1 \) are invariant.

Also \( y = 0 \) is an invariant line.
The picture confirms that $(0,0)$ is a stable node, and $(\pm 1,0)$ are saddles, as expected from the linearization!

- **THE EFFECT OF SMALL NONLINEAR TERMS - part II -**

  Consider this system
  \[
  \begin{align*}
  \dot{x} &= -y + ax(x^2+y^2) \quad \forall x \in \mathbb{R}, \\
  \dot{y} &= x + ay(x^2+y^2)
  \end{align*}
  \]

  Show that the linearized system incorrectly predicts that the origin is a center for all values of $a \in \mathbb{R}$.

  - $(0,0)$ EQUILIBRIUM.
    \[
    J_X(x,y) = \begin{pmatrix}
    3ax^2 + ay^2 & -1 + \cdots \\
    1 + \cdots & -
    \end{pmatrix}
    \]

    \[
    J_X(0,0) = \begin{pmatrix}
    0 & -1 \\
    1 & 0
    \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm i
    \]

    $(0,0)$ is a center for the linearized terms with $x$ and $y$. 

    \[
    J_Y = \begin{pmatrix}
    3ax^2 + ay^2 & -1 + \cdots \\
    1 + \cdots & -
    \end{pmatrix}
    \]

    $(0,0)$ is a center for the linearized terms with $x$ and $y$. 

    \[
    J_Y(0,0) = \begin{pmatrix}
    0 & -1 \\
    1 & 0
    \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm i
    \]
system around the origin!

Now we analyse the non-linear one.

We change variables to polar coordinates.

\[
\begin{cases}
    x = r \cos \theta \\
    y = r \sin \theta
\end{cases}
\]

\[
x^2 + y^2 = r^2 \implies x \dot{x} + y \dot{y} = 2z
\]

Equation for \( z = \ldots \):

\[
\dot{z} = x \left( -y + ax \left( x^2 + y^2 \right) \right) + y \left( x + ay \left( x^2 + y^2 \right) \right)
\]

\[
= -xy + ax^2 (x^2 + y^2) + xy + ay^2 (x^2 + y^2)
\]

\[
= a(x^2 + y^2)^2 = a \dot{r}^2 \dot{\theta}
\]

\[
\dot{z} = a \dot{r}^2 \dot{\theta}
\]

Equation for \( \theta ?! \) \( \theta = \ldots \):

\[
y \dot{\theta} = \frac{x \sin \theta}{x \cos \theta} = \tan \theta = 0 \implies \theta = \arctan \left( \frac{y}{x} \right)
\]

\[
\dot{\theta} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \frac{y \dot{x} - x \dot{y}}{x^2} = \frac{a \dot{r}^2}{x^2}
\]
\[
\frac{x}{y^2 + x^2} \cdot \frac{x [x + ay (x^2 + y^2)] - y/x}{x^2}
\]

\[
= \frac{1}{z^2} \cdot \frac{x^2 + ayx^3 + axy^3 + y^2 - ax^3 - ayx^3}{2^2} = 1 \rightarrow \text{a rotation!}
\]

\[
\begin{align*}
\dot{r} &= az^3 \\
\dot{\theta} &= 1
\end{align*}
\]

(\forall > 0)

\[
\text{stable spiral}
\]

\[
\text{unstable spiral}
\]

Centers of the linearized system are delicate!!
Def. \( \bar{z} \in \mathbb{R}^n, \ x(\bar{z}) = 0. \) 
\( \bar{z} \) eq. is called
• Hyperbolic if every eigenvalue of 
  \( A = \frac{\partial x(\bar{z})}{\partial \bar{z}} \) has real part different from 0. (This is the case of equilibrium \((0,0),(\pm i,0)\) in Ex. 1)
• Elliptic if all eigenvalues of 
  \( A = \frac{\partial x(\bar{z})}{\partial \bar{z}} \) have zero real part
  (but they are not zero!)
  (This is the case of \((0,0)\) in Ex. 2)

\( \downarrow \) For hyp. equilibria, the corresponding linearization well characterized the local linear system around them.

\( \downarrow \) "Grobman-Hartman theorem"

\( \hline \)