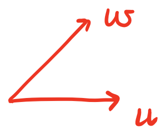
Lesson 7 - 12/10/2022

Thank you to the student who corrected me on the direction of rotation! I made a trivial mistake!

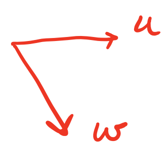
Recall the formula (in the case of 2 complex eigenvalues $\alpha \pm i\beta$):

$$\boxed{\varphi_t(z_0) = 2\rho\, e^{t\alpha} \left[\cos(\varphi + t\beta)u - \sin(\varphi + t\beta)w \right]}$$

- Suppose 

then the rotation direction is clockwise

[If $\varphi_0(z_0) \in \langle u \rangle$ then, for $t>0$, $\cos(\varphi+t\beta)$ has sign + but $\sin(\varphi+t\beta)$ has sign - !!]

- with the same argument, when 

then the rotation direction is counterclockwise

$\boxed{\text{TODAY : EXERCISES}}$

Date for the first written exam

11 Friday / 11 (10:30)

(in the morning)

IN <u>TORRE ARCHIMEDE</u> → Math Dep.

For ex. → • Repeat the ones done in class.

• Try to solve some ex. from STROGATZ book. → (Link)

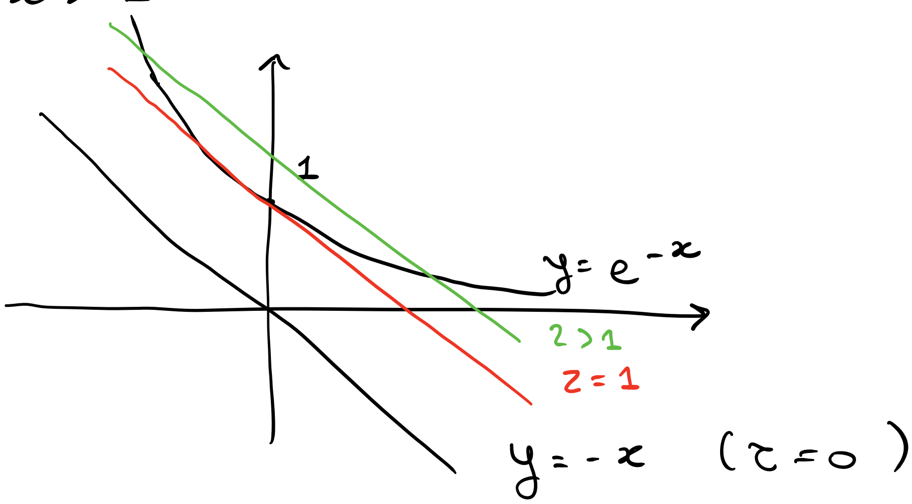
EX 1 Draw the bif. diag. for:

$$\dot{x} = \tau - x - e^{-x}, \qquad x \in \mathbb{R}, \ \tau \in \mathbb{R}$$

Sol
$$\underset{\overset{\|}{X_\tau(x)}}{X(x)} = \tau - x - e^{-x} = 0$$

$$\Leftrightarrow \quad \tau - x = e^{-x} \quad \longrightarrow$$ We adopt a geometric approach to det. (not explicitly) equilibria.

$$X(x) = \tau - x - e^{-x} > 0 \quad \Leftrightarrow \quad \tau - x > e^{-x}$$

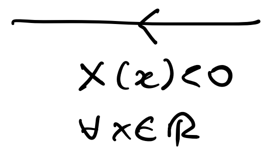


We need to det. the bif. case $\Rightarrow$ we need to impose that the graphs of $\tau - x$ and e^{-x} intersect tangentially.

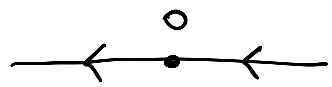
$$\begin{cases} e^{-x} = \tau - x \\ -e^{-x} = -1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \tau = 1 \\ x = 0 \end{cases}$$

Then, the following 3 cases occur:

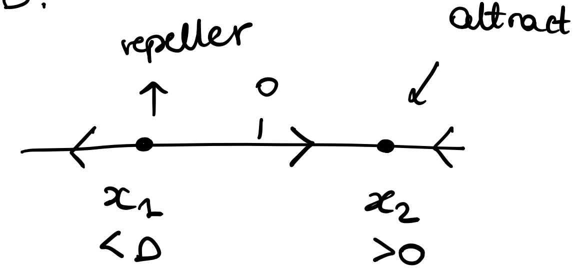
- $\lambda \in]-\infty, 1[$: NO EQUILIBRIA



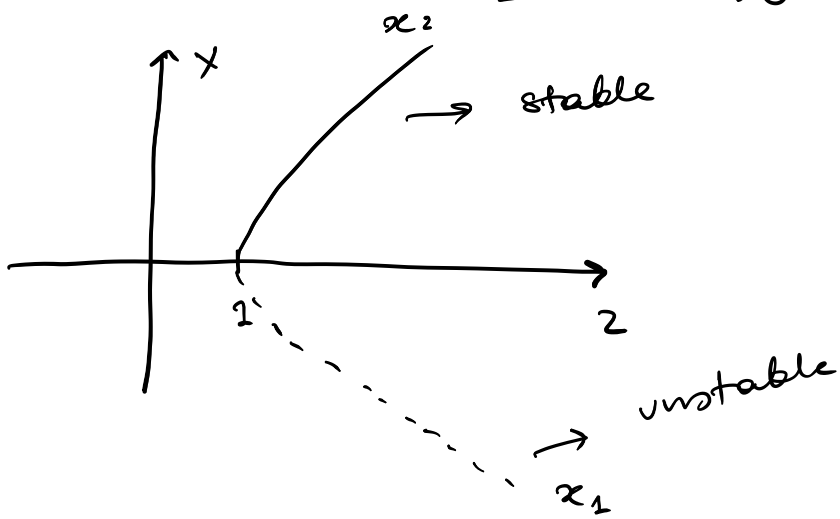
- $\lambda = 1$: $\exists!$ EQUILIBRIUM $x=0$
  and $X(x) < 0$, $x \neq 0$.



- $\lambda > 1$: 2 EQUILIBRIA.



Bif. diagram.



EX2

$$\dot{x} = -x + \beta \tanh x$$

where $x \in \mathbb{R}$, $\beta > 0$.

Sol

$$X(x) = -x + \beta \tanh x$$
$$\parallel$$
$$X_\beta(x)$$

$X(x) > 0 \iff x < \beta \tanh x$

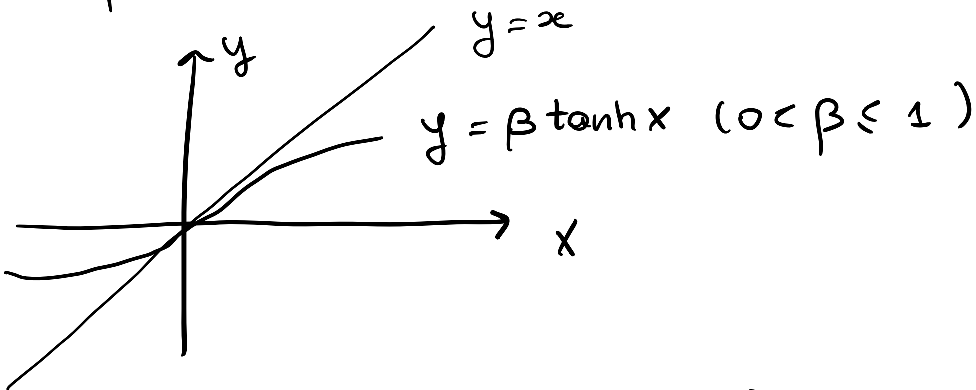
We use the same arguments of the previous ex.

$$\begin{cases} x = \beta \tanh x \\ 1 = \beta \dfrac{1}{\cosh^2 x} \end{cases} \iff \begin{cases} x = 0 \\ \beta = 1 \end{cases}$$

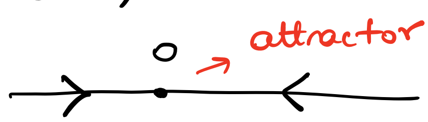
[or by recalling that $\tanh x = x + o(x) \ldots$]

So the next three cases occur:

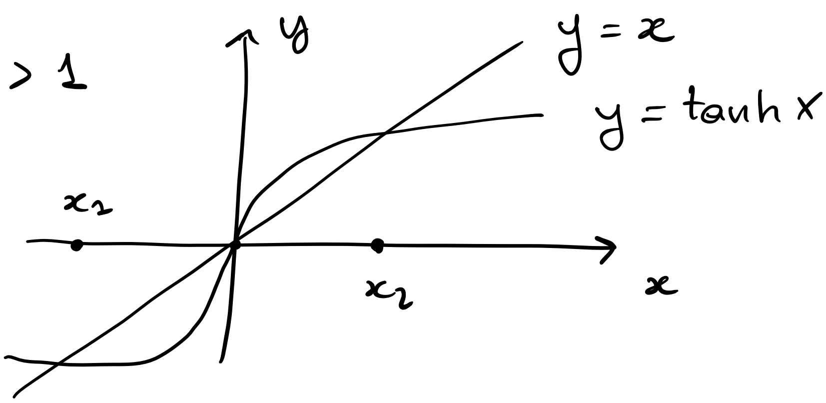
$0 < \beta \leq 1$



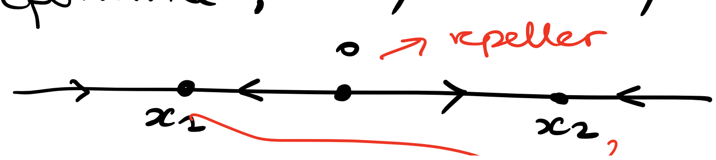
1 equilibrium, $x = 0$, $X(x) = -x + \beta \tanh x < 0$ for $x > 0$

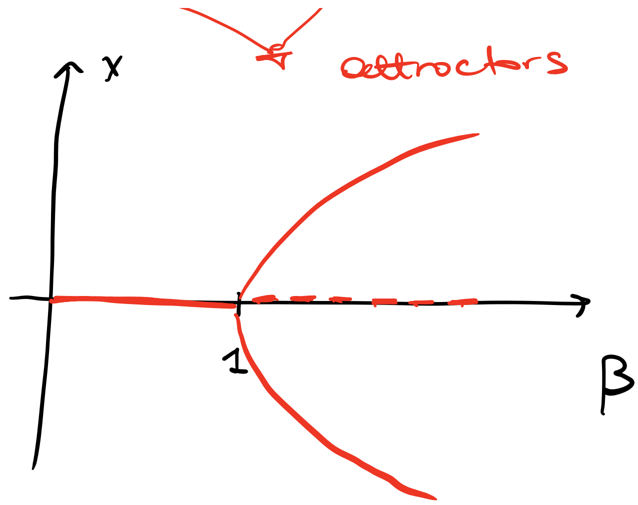


$\beta > 1$



3 equilibria, 0, $x_1 < 0$, $x_2 > 0$



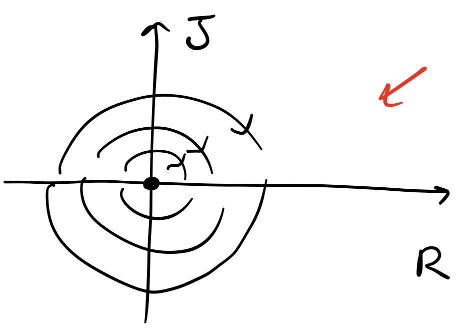


Ex 3 Love affairs! (I)

$$\begin{cases} \dot{R} = aJ \\ \dot{J} = -bR \end{cases} \qquad a, b > 0$$

$$A = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \rightarrow \lambda_{1,2} = \pm i\sqrt{ab}$$

$\rightarrow$ $(0,0)$ (the unique equilibrium is a center)



A neverending cycle of love and hate!

Ex 4 Love affairs! (II)

$$\begin{cases} \dot{R} = aR + bJ \\ \dot{J} = bR + aJ \end{cases}$$ where $a < 0$, $b > 0$

This model means that – in such a case – Romeo and Juliet are "cautious" lovers!

In particular:

$a < 0$ measures the rate of cautiousness.

$b > 0$ measures the rate of responsiveness.

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Eigenvalues are given by $\lambda^2 - 2a\lambda + a^2 - b^2 = 0$

$\Leftrightarrow \lambda_{1,2} = a \pm \sqrt{b^2} = a \pm b$

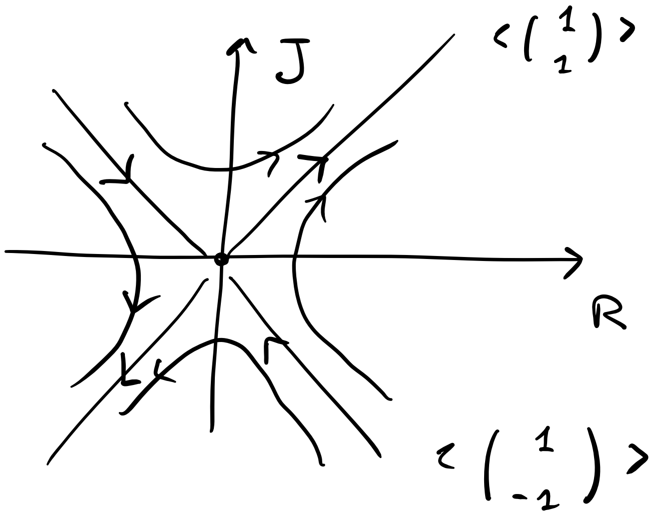
$\lambda_1 = a + b$ has eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = a - b$ has eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

FIRST CASE

$a - b < 0$ always.

$a + b > 0 \Leftrightarrow b > -a$ $\Rightarrow$ we obtain a saddle!

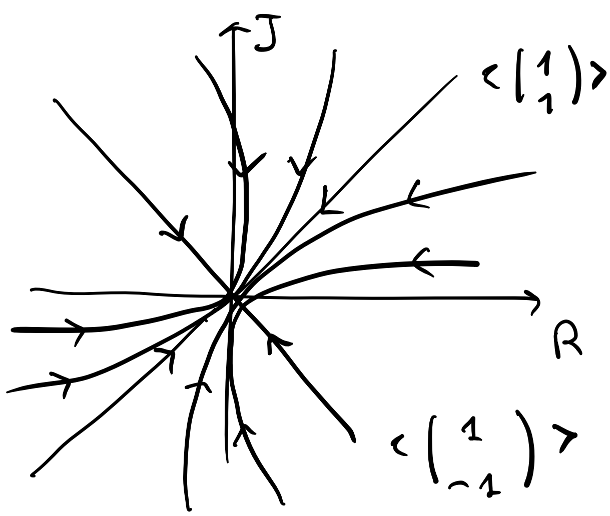


SECOND CASE

$a-b<0$ always
$a+b<0 \iff b<-a$

$\iff (0,0)$ is a stable node

$\Longrightarrow$ The relation always finishes in mutual indifference!



[We don't analyse the case $b=-a$
$\rightarrow A = a\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, not diag.]

• THE EFFECT OF SMALL NONLINEAR TERMS - part I -

• Find equilibria for the system

$$\begin{cases} \dot{x} = -x + x^3 \\ \dot{y} = -2y \end{cases}$$

and use linearization to classify them.
Then check that the conclusions on the linear systems to draw the phase portrait for the full non linear case.

• $\begin{cases} -x + x^3 = x(-1 + x^2) = 0 \\ y = 0 \end{cases}$

$\swarrow$

$x = 0, \quad x = \pm 1$

$P = (0,0), \ (1,0), \ (-1,0)$

$$JX(x,y) = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$JX(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow (0,0) \text{ is a stable node}$$

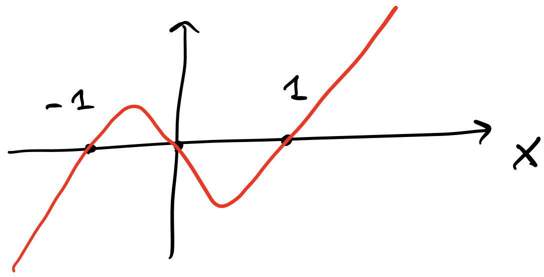
$$JX(\pm 1, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow (\pm 1, 0) \text{ are both saddle points.}$$

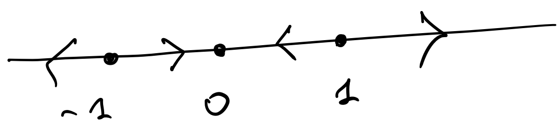
But, in such a case, we can also

check explicity the behaviour of solutions for the simplified non-linear system, since eqs. are uncoupled.

$\dot{y} = -2y \longrightarrow$ In the y-direction, all trajectories decay exp. to 0.

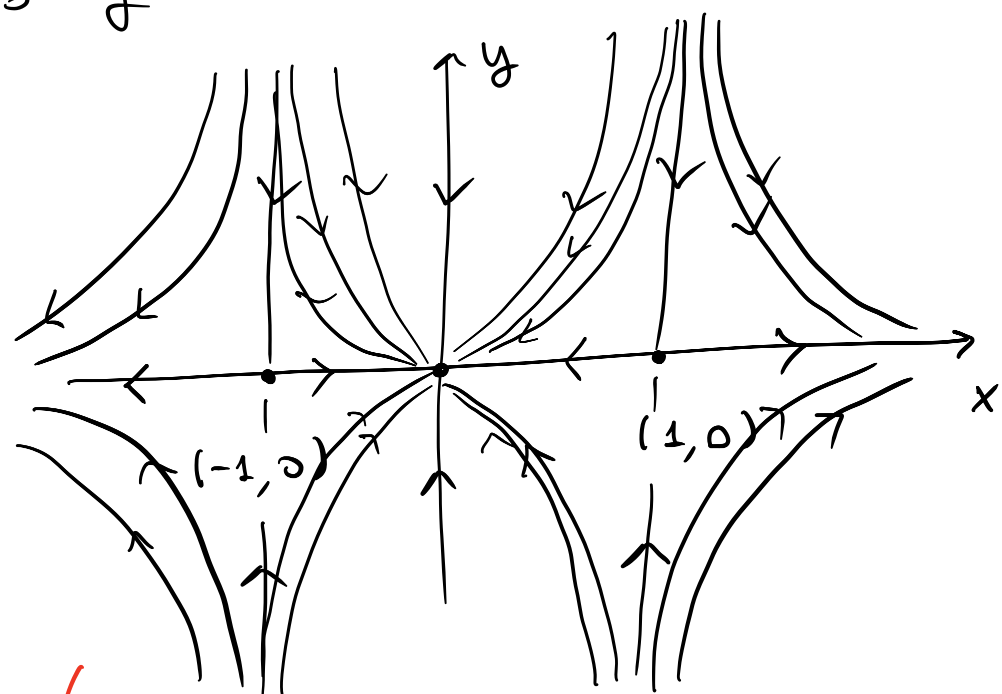
$\dot{x} = x(-1+x^2)$





For the system on the plane, the lines $x=0$ and $x=\pm 1$ are invariant.

Also $y=0$ is an invariant line.



The picture confirms that $(0,0)$ is a stable node, and $(\pm 1, 0)$ are saddles, as expected from the linearization!

• THE EFFECT OF SMALL NONLINEAR TERMS – part II –

• Consider this system

$$\begin{cases} \dot{x} = -y + a x (x^2+y^2) \\ \dot{y} = x + a y (x^2+y^2) \end{cases} \qquad a \in \mathbb{R}.$$

Show that the linearized system <u>incorrectly</u> predicts that the origin is a center for all values of $a \in \mathbb{R}$.

• $(0,0)$ EQUILIBRIUM.

terms with x and y.

$$Jx(x,y) = \begin{pmatrix} 3ax^2 + ay^2 & -1 + \dots \\ 1 + \dots & \dots \end{pmatrix}$$

$\Downarrow$

$$Jx(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm i$$

$(0,0)$ is a center for the linearized

system around the origin!

Now we analyse the non linear one.

We change variables to POLAR COORDINATES.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$x^2+y^2 = r^2 \Rightarrow x\dot{x} + y\dot{y} = r\dot{r}$$

Equation for $\dot{r} = \dots$

$$\cancel{r}\dot{r} = x\underbrace{\left(-y + ax(x^2+y^2)\right)}_{\dot{x}} + y\underbrace{\left(x + ay(x^2+y^2)\right)}_{\dot{y}}$$

$$= \cancel{-xy} + ax^2(x^2+y^2) + \cancel{xy} + ay^2(x^2+y^2)$$

$$= a(x^2+y^2)^2 = a\cancel{r^4}\, r^3 \quad \boxed{r>0}$$

$$\dot{r} = ar^3$$

Equation for θ?! $\dot{\theta} = \dots$

$$\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \operatorname{tg}\theta \Rightarrow \theta = \operatorname{arctg}\left(\frac{y}{x}\right)$$

$$\dot{\theta} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\dot{y}x - \dot{x}y}{x^2} =$$

$$\frac{\dot{y}}{}$$

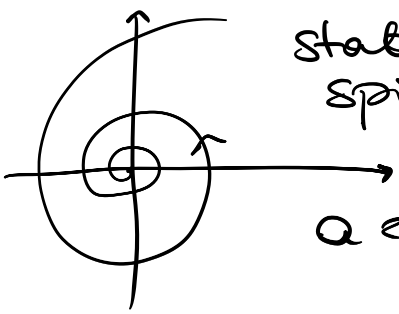
$$= \frac{\cancel{x^2}}{y^2+x^2} \cdot \frac{x[\dot{x} + a y (x^2+y^2)] - y(x)}{\cancel{x^2}}$$

$$\underset{r^2}{}$$

$$= \frac{1}{r^2} \cdot x^2 + \cancel{a y x^3} + \cancel{a x y^3} + y^2 - \cancel{a x^3 y} - \cancel{a x y^3}$$

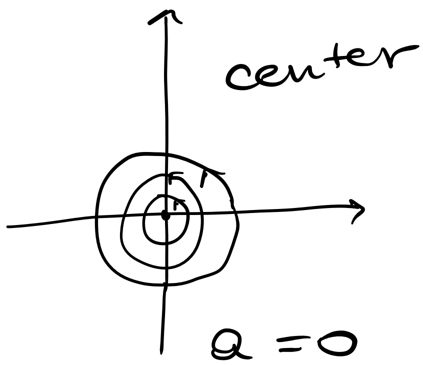
$$= \frac{(x^2+y^2)}{r^2} = \frac{r^2}{r^2} = 1 \longrightarrow \text{a rotation!}$$

$$\begin{cases} \dot{r} = a r^3 \\ \dot{\theta} = 1 \end{cases} \qquad (r>0)$$



stable spiral

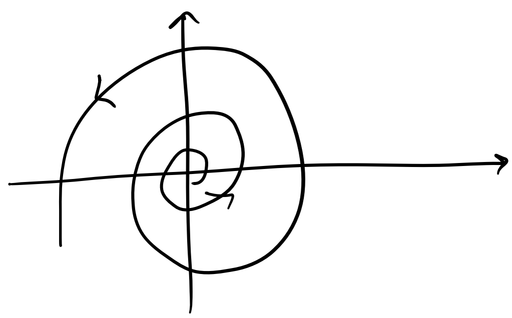
$a<0$



center

$a=0$

unstable spiral



$a>0$

Centers of the linearized system are delicate!!

$\boxed{\text{Def}}$ $\bar{z} \in \mathbb{R}^n$, $X(\bar{z}) = 0$.

$\bar{z}$ eq. is called

- HYPERBOLIC if every eigenvalue of $A = \frac{\partial X}{\partial z}(\bar{z})$ has real part different from 0. (THIS IS THE CASE OF EQUILIBRIA $(0,0)$, $(\pm 1, 0)$ IN EX. 1)

- ELLIPTIC if all eigenvalues of $A = \frac{\partial X(\bar{z})}{\partial z}$ have zero real part (but they are not zero!) (THIS IS THE CASE OF $(0,0)$ IN EX. 2)

For hyp. equilibria, the corresponding linearization well characterized the non linear system around them.

"Grobman-Hartman theorem"

—x—x—