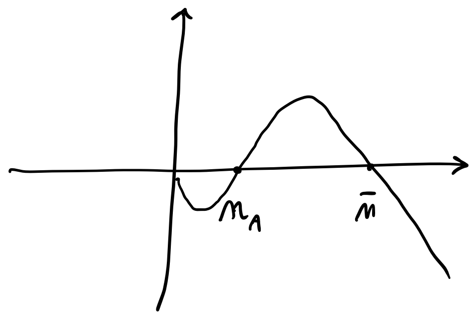
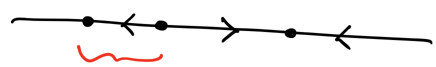
Lesson 3 - 03/10/2022

Allee effect : a correction of the logistic model

$$\left(\dot{x} = K\left(1 - \frac{x}{\bar{m}}\right) x \right)$$

$$\dot{x} = K\left(1 - \frac{x}{\bar{m}}\right)\underbrace{\left(\frac{x}{m_A} - 1\right)}_{0 < m_A < \bar{m}} x$$



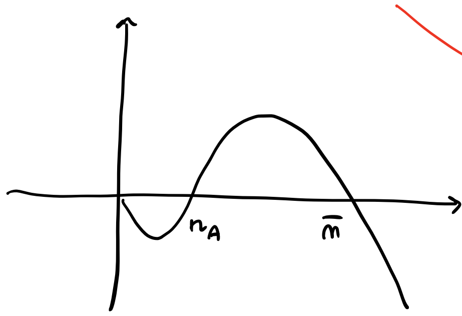


the population decreases also if it becomes too small and sparse.

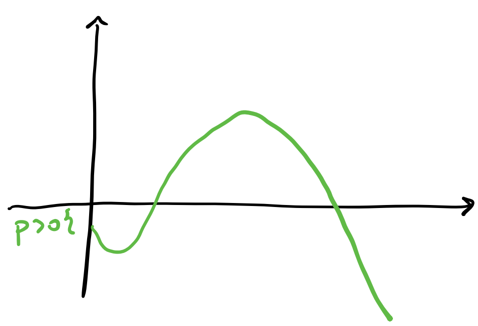
Which the effect of a "constant hunting parameter" to a population with a dynamics given by ??

$$\dot{x} = K\left(1 - \frac{x}{\bar{m}}\right)\left(\frac{x}{m_A} - 1\right) x \underbrace{- p}_{\text{term of "constant hunting"}} \qquad (p > 0)$$

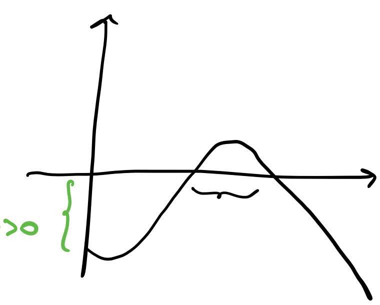
V.F. depending on a (real) parameter.

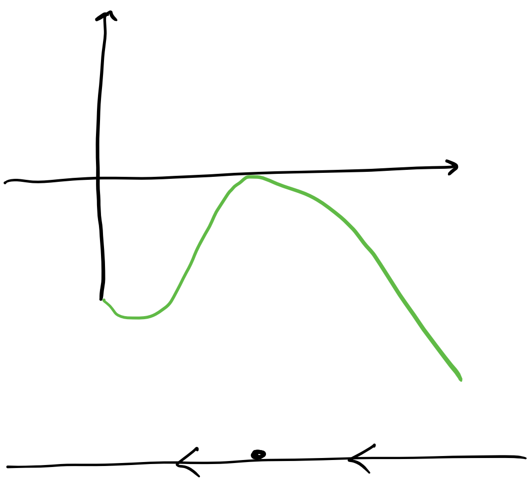


$p = 0$



$p>0$





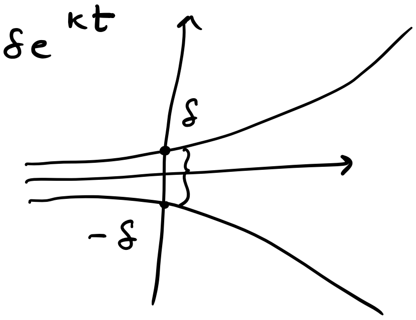
$$p = \max_{x \in \mathbb{R}} X(x)$$

From the next lesson: Bifurcations $\rightarrow$ dependence of equilibria and their stability in terms of a parameter.

Now: $\boxed{\text{dependence on initial data}}$

1. $\dot{x} = kx$, $k > 0$, $x \in \mathbb{R}$

$$\varphi^t(x_0) = x_0 e^{kt} \Rightarrow \begin{cases} \phi^t(\delta) = \delta e^{kt} & \delta > 0 \\ \phi^t(-\delta) = -\delta e^{kt} \end{cases}$$



$$|\varphi^t(\delta) - \varphi^t(-\delta)| = 2\delta e^{kt} < \varepsilon$$

$$\Leftrightarrow \delta < \frac{\varepsilon}{2} e^{-kt}$$

If we want to make previsions with foresight, we need to know the initial data very carefully.

GENERAL THEOREM

$X \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, complete.

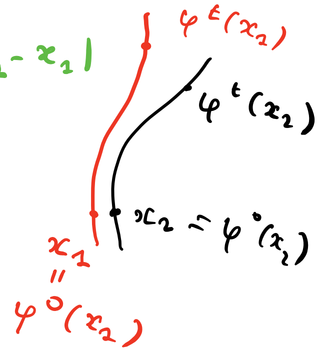
X globally Lip-continuous with Lip. constant $K > 0$.

That is:

$$|X(x_2) - X(x_1)| \le K |x_2 - x_1| \quad \forall\, x_1, x_2 \in \mathbb{R}^n$$

Then

$$e^{-k|t|}|x_2-x_1| \leqslant |\varphi^t(x_2)-\varphi^t(x_1)| \leqslant e^{k|t|}|x_2-x_1|$$



At best, trajectories converge as $e^{-k|t|}$

At worst, trajectories diverge as $e^{k|t|}$

(Malthusian model)

$\forall x_2, x_1 \in \mathbb{R}^2$
$\forall t \in \mathbb{R}$

$\boxed{\text{Proof}}$

$\varphi^t(x_2)$, $\varphi^t(x_1)$ are such that
$$\begin{cases} \dot{\varphi}^t(x_2) = X(\varphi^t(x_2)) \\ \dot{\varphi}^t(x_1) = X(\varphi^t(x_1)) \end{cases}$$

Strategy: estimate the derivative of δ_t^2 where $\delta_t = |\varphi^t(x_2) - \varphi^t(x_1)|$

Remark $\delta_t > 0$ since (X is Lip-cont.) traj. cannot intersect !!

$$\frac{d}{dt}\delta_t^2 = 2(\varphi^t(x_2) - \varphi^t(x_1))\cdot(\dot{\varphi}^t(x_2) - \dot{\varphi}^t(x_1))$$
$$= 2(\varphi^t(x_2) - \varphi^t(x_1))\cdot(X(\varphi^t(x_2)) - X(\varphi^t(x_1)))$$

Therefore

$$\left|\frac{d}{dt}\delta_t^2\right| \leqslant 2|\varphi^t(x_2) - \varphi^t(x_1)||X(\varphi^t(x_2)) - X(\varphi^t(x_1))|$$
$$\leqslant 2\,\overbrace{|\varphi^t(x_2) - \varphi^t(x_1)|\,K\,|\varphi^t(x_2) - \varphi^t(x_1)|}^{\delta_t^2}$$

Lip. property

$$= 2K\delta_t^2$$

$$\Rightarrow \left|\frac{d}{dt}\delta_t^2\right| \leqslant 2K\delta_t^2$$

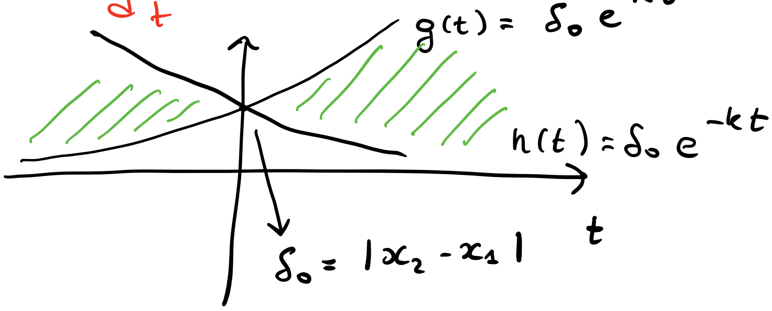
Moreover:

$$\frac{d}{dt}\delta_t^2 \underset{\text{By def.}}{=} 2\delta_t \frac{d\delta_t}{dt} \Rightarrow \cancel{2}\cancel{\delta_t}\left|\frac{d}{dt}\delta_t\right| \leq \cancel{2}k\delta_t^{\cancel{2}}$$

$$\underset{\delta_t>0}{\Leftrightarrow} \left|\frac{d}{dt}\delta_t\right| \leq k\,\delta_t$$

$$\Leftrightarrow \boxed{-k\delta_t \leq \frac{d}{dt}\delta_t \leq k\delta_t}$$

As a consequence, the graph of δ_t is between $g(t) = \delta_0 e^{kt}$ and $h(t) = \delta_0 e^{-kt}$ satisfying respectively $\frac{d}{dt}g(t) = kg(t)$ and $\frac{d}{dt}h(t) = -k\,h(t)$



That is

$$e^{-k|t|}|x_2-x_1| \leq |\varphi^t(x_2)-\varphi^t(x_1)| \leq e^{k|t|}|x_2-x_1|$$

$\forall\, x_1, x_2 \in \mathbb{R}^n$, $\forall\, t\in\mathbb{R}$.

$\downarrow$

The divergence / convergence <u>CAN</u> be (not MUST be) exponential. ☒

("deterministic chaos")

$\boxed{1}$ Malthusian model : exponential divergence of solutions starting close to 0.

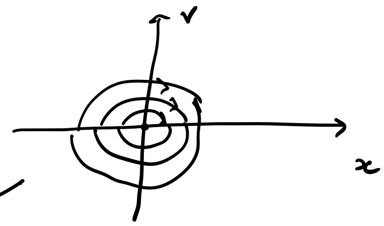
$\boxed{2}$ $\dot{x} = 1 \to \varphi^t(x_0) = x_0 + t$

$\to |\varphi^t(x_1) - \varphi^t(x_0)| = |x_1 - x_0|$

$\to$ constant separation of solutions.

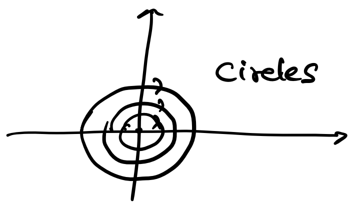
$\boxed{3}$ Harmonic oscillator

$\ddot{x} = -\omega^2 x$



bounded separation of solutions.

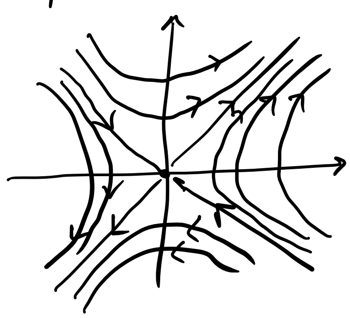
$\boxed{\omega = 1}$ $(k = m)$



Circles $\to$ constant sep. of solutions.

$\boxed{4}$ Harmonic repeller.

It can be proved that there is exponential divergence of solutions.



General remarks - Strong sensitivity to initial conditions -

$X \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, complete and globally Lip $(k > 0)$

Supp. that separations of solutions is exponential for $t > 0$.

$|\varphi^t(x_1) - \varphi^t(x_0)| \sim e^{kt} |x_1 - x_0|$

Then, if I impose $|\varphi^t(x_1) - \varphi^t(x_0)| \sim \varepsilon$,

I obtain

$$e^{kt} |x_1 - x_0| \sim \varepsilon$$

that $|x_1 - x_0| \sim \varepsilon e^{-kt}$

(

↓ It is therefore impossible to make previsions for long time since there is a strong sensitivity of initial conditions!!

accurate

Determinism of diff. eqs. $\longleftrightarrow$ Exponential divergence of solutions also with close initial data.

Predictability of Cauchy theorem $\longleftrightarrow$ strong sensibility to initial conditions.

Dependence on parameters

We can give same conclusions of the dependence on initial data! In fact

$$X \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^k; \mathbb{R}^n)$$

↓

k parameters

This is a v.f. dep. on $(\mu_1 - \mu_k) = \mu$ parameters.

We can consider the "extension" of the v.f. to the parameters by
$$\begin{cases} \dot{x} = X(x, \mu) \\ \dot{\mu} = 0 \end{cases}$$

$\Rightarrow$ the dependence on parameters follows the same conclusions as for initial data (and then can be at worst exponential).

↓

Bifurcations

$\boxed{1}$ $\dot{x} = \mu$ $\quad X(x) = \mu.$ $\quad x, \mu \in \mathbb{R}$

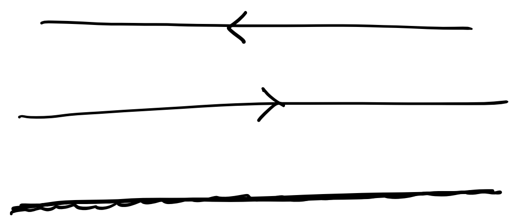
$x(t; 0, x_0, \mu) = x_0 + \mu t$

Phase portraits

$\mu < 0$

$\mu > 0$

$\mu = 0$

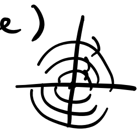


fixed points $X(x) = 0$

$\boxed{2}$ $\ddot{x} = \mu x$ $\quad (x, \mu \in \mathbb{R})$

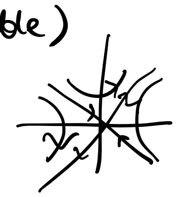
$$\begin{cases} \dot{x} = v \\ \dot{v} = \mu x \end{cases} \qquad X(x, v; \mu) = (v, \mu x)$$

$\mu < 0 \to$ harmonic oscillator (one equilibrium, stable) $= (0,0)$

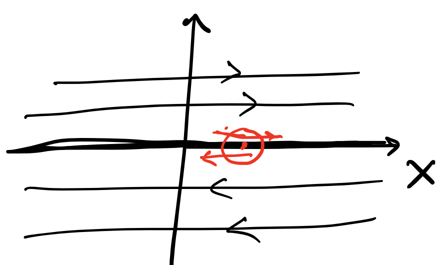


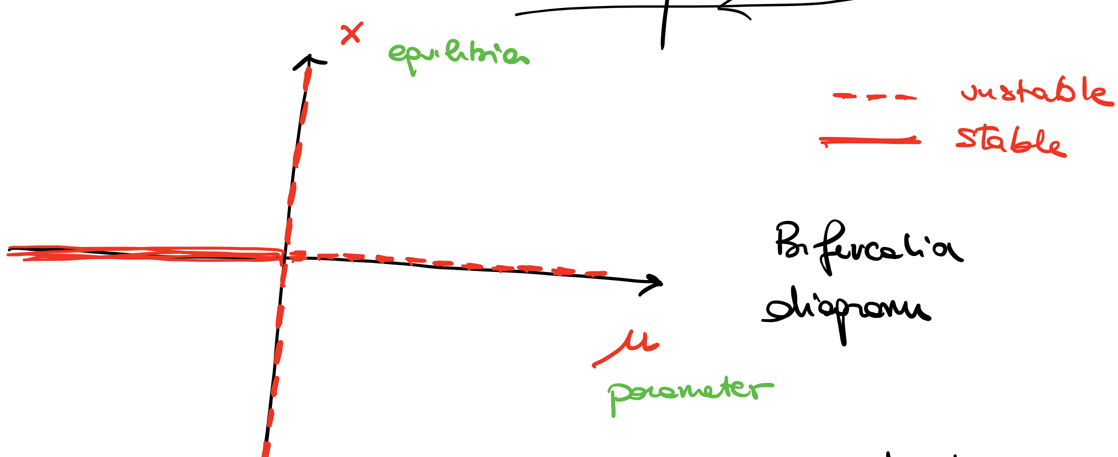
$\mu = 0 \to$ every point is an equilibrium (unstable fixed point) on the x-axis

$\mu > 0 \to$ harmonic repeller (one equilibrium, unstable) $= (0,0)$



free particle





Bifurcation diagram

Attention X here is not the

equilibrium but the equilibrium configuration !

$(x, 0)$

$\downarrow$ equilibrium !