Deep Inelastic Scattering

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- Electron-Proton scattering. Proton Form Factors
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- Relationship between DIS and $\sigma_{tot}(\gamma^{*}p)$
- Naive Quark-Parton Model:
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- Improved Parton Model
- Scaling violation. Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation (DGLAP)
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Probing a Charge Distribution with Electrons

- Method: to measure the angular distribution of the scattered e^- and compare it to the known cross section for scattering e^- from a point charge:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{point} \cdot \left|F(q)^2\right|$$

$q = k_i - k_f \equiv$ momentum trasfer between the incident e^- and the target

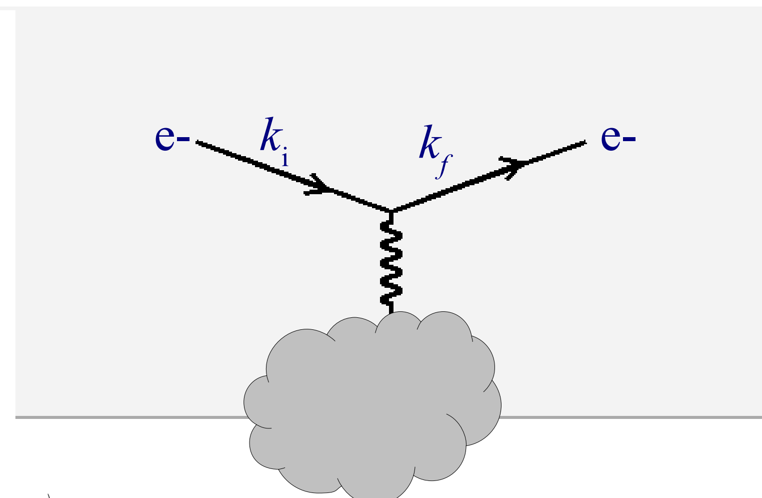
- **Scattering of non polarized e^- with energy E from a static, spinless charge distribution** $Ze\,\rho(\vec{x})$ **:** One has:

$$\int \rho(\vec{x})\, d^3x = 1$$

The form factor is (for a static target $\equiv M = \infty$):

$$F(\vec{q}) = \int \rho(\vec{x})\, e^{i\vec{q}\cdot\vec{x}}\, d^3x$$

The cross-section for a static and structureless target is:



$$\left(\frac{d\sigma}{d\Omega}\right)_{point} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{(Z\alpha)^2 E^2}{4k^4 \sin^4\theta/2}\left(1 - v^2 \sin^2\theta/2\right)$$

$k = |\vec{k}_i| = |\vec{k}_f|; \quad v = k/E; \quad \theta =$ scattering angle

Probing a Charge Distribution with Electrons

- From the normalization condition:

$$\int \rho(\vec{x})\, d^3x = 1$$

and from:

$$F(\vec{q}) = \int \rho(\vec{x})\, e^{i\vec{q}\cdot\vec{x}}\, d^3x$$

one obtains:

$$F(0) = \int \rho(\vec{x})\, d^3x = 1$$

If q is not too large, it is possible to expand the exponential:

$$\begin{aligned} F(\vec{q}) &= \int \rho(\vec{x})\, e^{i\vec{q}\cdot\vec{x}}\, d^3x = \int \rho(\vec{x}) \left[1 + i\vec{q}\cdot\vec{x} - \frac{(\vec{q}\cdot\vec{x})^2}{2} + \ldots \right] d^3x \\ &= 1 + \int \rho(\vec{x})\, i\vec{q}\cdot\vec{x}\, d^3x - \frac{1}{2}\int \rho(\vec{x}) |\vec{q}|^2 x^2\, d^3x + \ldots \\ &= 1 - \frac{1}{6}|\vec{q}|^2 \langle r^2 \rangle + \ldots \end{aligned}$$

assuming that ρ is spherically symmetric.

Probing a Charge Distribution with Electrons

That is for small q it is possible to measure the $<r^2>$ of the charge cloud. It is not sensitive to the detailed structure.
In the limit of small q the photon is soft and with its large wavelength cannot solve the details.

If instead:

$$\rho(r) \propto e^{-mr}$$

one has that:

$$F(\vec{q}) = \left(1 - \frac{q^2}{m^2}\right)^{-2}$$

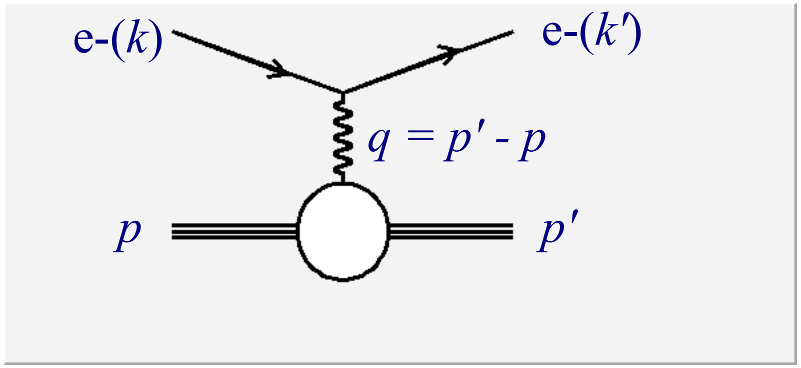
Electron-Proton Scattering. Proton Form Factors

- If one wants to explore the **proton structure function**:
  - there is also the magnetic moment of the proton, not only its charge
  - the proton is not static, but will recoil under the electron's bombardment
- In the case the **proton** was a **point charge**: charge: e^+, Dirac magnetic moment: $e/2M$.
  One can make the identification: $e^- p \rightarrow e^- p \equiv e^- \mu^- \rightarrow e^- \mu^-$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha^2}{4E^2 \sin^4\theta/2}\right)\frac{E'}{E}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

from the recoil of the target: $\dfrac{E'}{E} = \dfrac{1}{1+\dfrac{2E}{M}\sin^2\theta/2}$

- Scattering amplitude:
  (taken from $e^- \mu^- \rightarrow e^- \mu^-$) $T_{fi} = -i\int j_\mu \left(\frac{-1}{q^2}\right) J^\mu d^4x$



$$j_\mu = -e\,\bar{u}(k')\gamma_\mu u(k)\, e^{i(k'-k)x}$$
$$J_\mu = +e\,\bar{u}(p')[\ldots]u(p)\, e^{i(p'-p)x}$$

$$[\ldots] \neq \gamma^\mu$$
$$[\ldots] = \left[F_1(q^2)\gamma^\mu + \frac{\xi}{2M}F_2(q^2)\, i\sigma^{\mu\nu}q_\nu\right]$$

the proton is not pointlike
terms with γ^5 are ruled out by the conservation of parity

Electron-Proton Scattering. Proton Form Factors

- F_1, F_2: independent form factors
  ξ = anomalous magnetic moment
- For $q^2 \to 0$ the proton has no structure, it is only a particle of charge e and magnetic moment $(1+\xi)e/2M$ con $\xi = 1.79$
  therefore:

$$F_1(0) = 1; \qquad F_2(0) = 1$$

  for a neutron:

$$F_1(0) = 0; \qquad F_2(0) = 1; \qquad \text{and } \xi = -1.91$$

- Making the calculation (**Rosenbluth formula**):

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha^2}{4E^2\sin^4\theta/2}\right)\frac{E'}{E}\left[\left(F_1^2 - \frac{\xi^2 q^2}{4M^2}F_2^2\right)\cos^2\theta/2 \;-\; \frac{q^2}{2M^2}\left(F_1 + \xi F_2\right)^2\sin^2\theta/2\right]$$

F_1, F_2 = parametrize our ignorance of the detailed structure of the proton
F_1, F_2 can be determined experimentally by measuring $d\sigma/d\Omega$ as function of θ and q^2

Electron-Proton Scattering. Proton Form Factors

- To eliminate interference terms such as $F_1 F_2$ one can define:

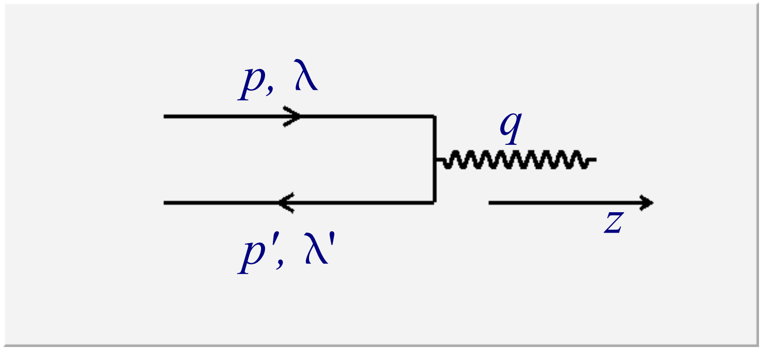
$$G_E \equiv F_1 + \frac{\xi q^2}{4M^2} F_2$$
$$G_M \equiv F_1 + \xi F_2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha^2}{4E^2 \sin^4\theta/2}\right)\frac{E'}{E}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\theta/2 \; + \; 2\tau G_M^2 \sin^2\theta/2\right]$$

$\tau \equiv -q^2/4M^2$

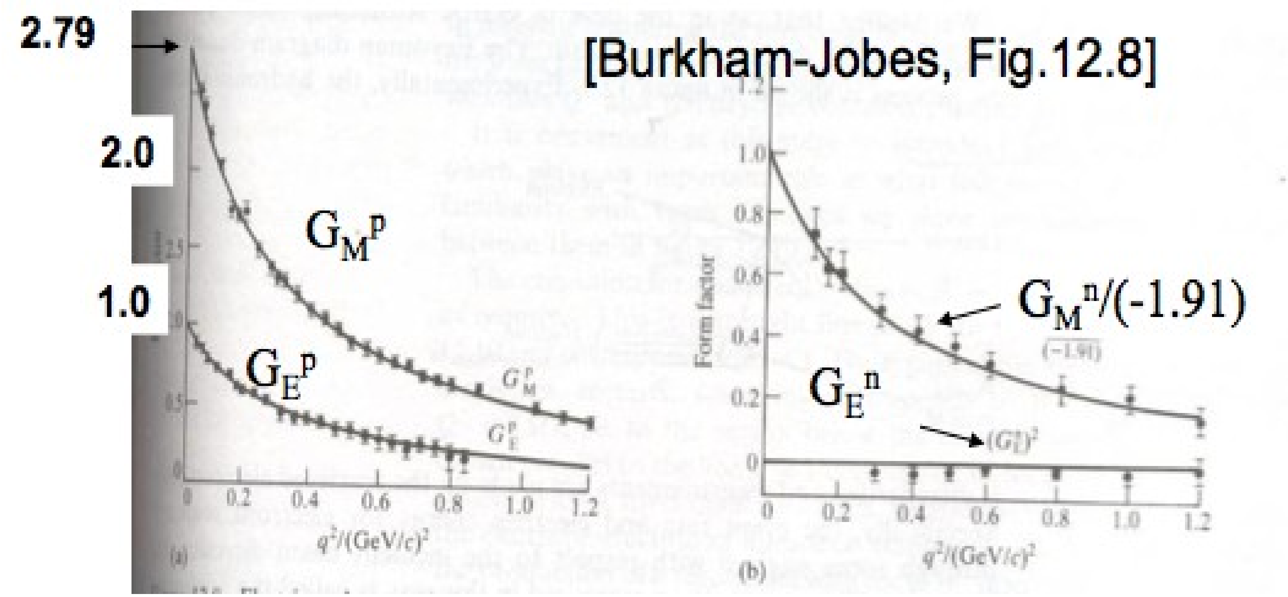
G_E, G_M generalization of $F(q^2)$.

they are connected to the charge distribution (G_E) and magnetic moment (G_M) in a particular reference frame called the **Breit** (or brick wall) frame defined by $\vec{p}\,' = -\vec{p}$
But in an arbitrary reference , the presence of the recoil of the proton destroys the link.
In the Breit frame there is no energy transfer to the proton.



Electron-Proton Scattering. Proton Form Factors

- For small q^2 ($|\boldsymbol{q}|^2 \ll M^2$) the form factors G_E, G_M are the Fourier transform of the charge and magnetic distributions of the proton.
- G_E, G_M are extracted from the angular distribution of $ep \rightarrow ep$



Electron-Proton Scattering. Proton Form Factors

- Fitting the data one has:

$$G_E(q^2)=\left(1-\frac{q^2}{0.71}\right)^{-2} \qquad \text{in units of GeV}^2$$

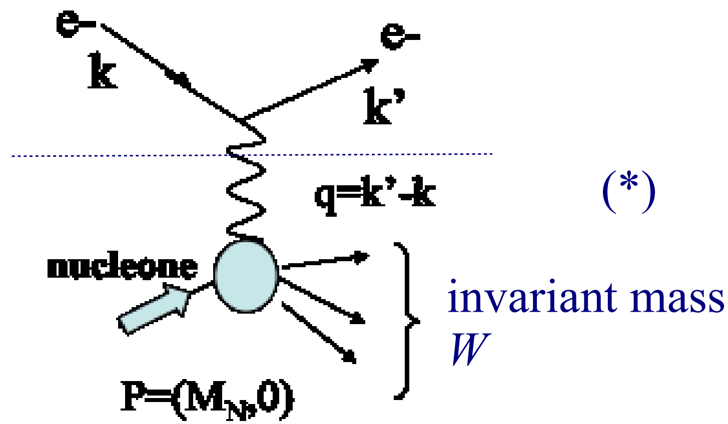
- For small q^2 the G_E can be used to calculate for example:

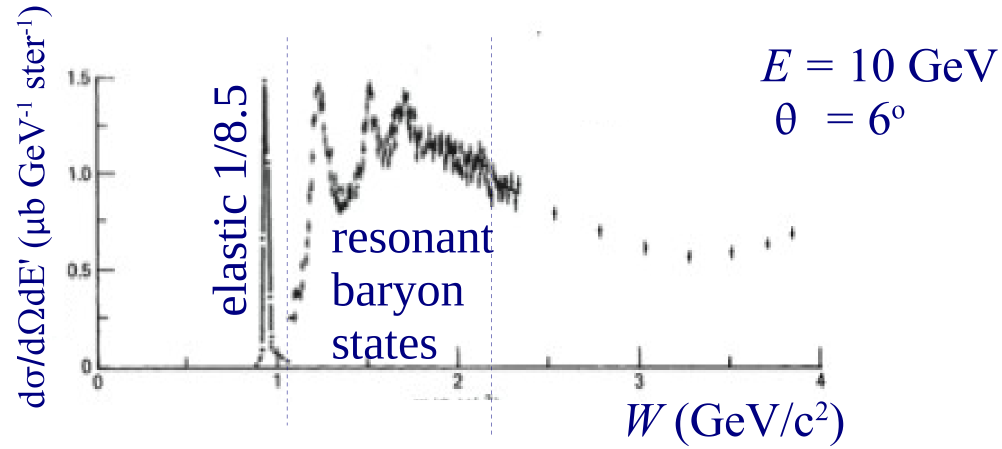
$$<r^2>=6\left(\frac{dG_E(q^2)}{d\,q^2}\right)_{q^2=0} = (0.81\cdot 10^{-15}\,\mathrm{m})^2$$

the same radius is obtained for the magnetic distribution.
The charge distribution of the nucleon has the shape: e^{-mr}

Inelastic Scattering *ep* → *eX*

- Increasing further $-q^2$ to see better the details of the proton:





For modest $-q^2$, the p becomes excited: $ep \rightarrow e\Delta^+ \rightarrow ep\pi^0$, etc.
When $-q^2$ is very large → the debris of the p becomes too messy → necessity of a new formalism
.

- Observing (*) the upper part remains unchanged respect to $ep \rightarrow ep$. The lower part has to be changed because in the final state there is not a single fermion but a rather complex structure.

$$d\sigma \sim L^e_{\mu\nu}\left(L^p\right)^{\mu\nu} \;\rightarrow\; d\sigma \sim L^e_{\mu\nu} W^{\mu\nu}$$

- $L^e_{\mu\nu} = \frac{1}{2}\sum_{spins}\left[\bar{u}(k')\gamma_\mu u(k)\right]\left[\bar{u}(k')\gamma_\nu u(k)\right]^*$ **leptonic tensor**

- $W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^\mu p^\nu + \frac{W_4}{M^2} q^\mu q^\nu + \frac{W_5}{M^2}\left(p^\mu p^\nu + q^\mu q^\nu\right)$ **hadronic tensor** (describing our ingnorance)

Inelastic Scattering $ep \rightarrow eX$

- $W_{\mu\nu}$ is built using $g_{\mu\nu}$ and the independent moments p, q ($p' = p + q$).
  γ^μ is not included because the cross section has already summed and avergared over the spins.
- In $W_{\mu\nu}$ there are no antisymmetric terms, they give no contribution after the insertion into the product because the tensor $L_{\mu\nu}$ is symmetric.
- W_3 is not present, it is reserved for the P violation, important in the case of DIS with neutrinos.
- It can be demonstrated that from:

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$$

one obtains:

$$W_5 = -\frac{p \cdot q}{q^2} W_2; \qquad W_4 = -\left(\frac{p \cdot q}{q^2}\right)^2 W_2 + \frac{M^2}{q^2} W_1$$

Only 2 of the 4 inelastic structure functions are independent:

$$W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2 \frac{1}{M^2}\left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right)$$

- W_i = functions of the Lorentz scalar variables that can be constructed from the 4-momenta at the hadronic vertex.

Inelastic Scattering $ep \rightarrow eX$

- Unlike elastic scattering, **in DIS there are two independent variables.** For example:

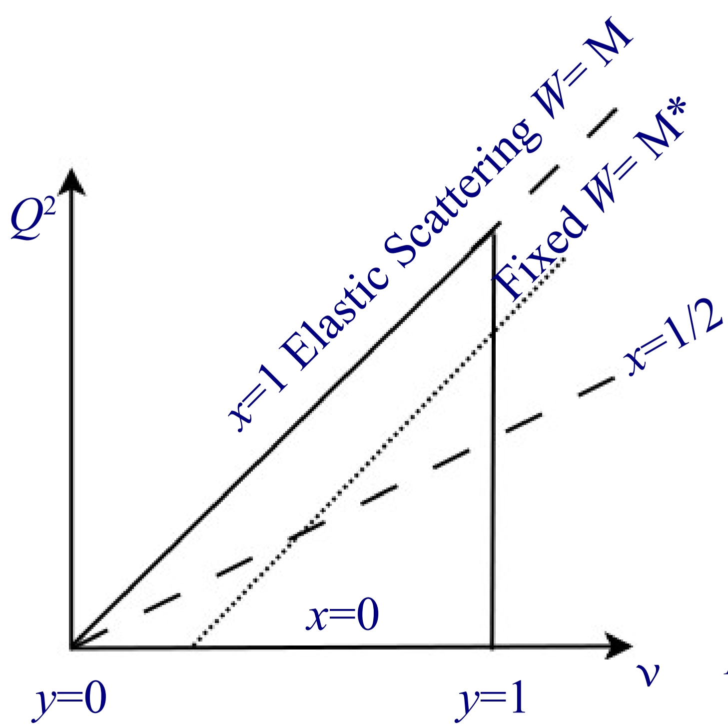
$$q^2; \quad \nu = \frac{p \cdot q}{M}$$

or:

$$x = \frac{-q^2}{2\, p \cdot q} = \frac{-q^2}{2 M \nu}; \qquad y = \frac{p \cdot q}{p \cdot k}$$

$$0 \leqslant x \leqslant 1; \qquad 0 \leqslant y \leqslant 1$$

Putting $Q^2 \equiv -q^2$



Triangle: kinematic region allowed per $ep \rightarrow eX$
Invariant mass W of the hadronic final system:

$$W^2 = (p+q)^2 = M^2 + 2M\nu + q^2$$

$\nu_{max} = E$ in the laboratory reference frame

Inelastic Scattering *ep* → *eX*

- In the reference system where the proton is at rest:

$$\nu = E - E', \qquad y = \frac{E - E'}{E}$$

Inelastic Scattering *ep* → *eX*

- On has:

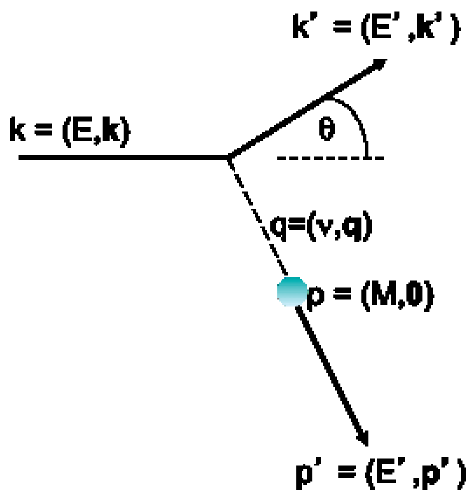
$$\frac{d\sigma}{dE' \, d\Omega} = \frac{\alpha^2}{q^4} \frac{E'}{E} \left(L^e\right)^{\mu\nu} W_{\mu\nu}$$

and making the calculations:

$$\left(\frac{d\sigma}{dE' \, d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \left(W_2(\nu, q^2) \cos^2 \theta/2 \; + \; 2W_1(\nu, q^2) \sin^2 \theta/2\right)$$

Inelastic Scattering *ep* → *eX*

- Summarizing and using the kinematics in the laboratory reference frame (fixed target):



general formula: $$\frac{d\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\{\ldots\}$$

- **μ-target of mass *m***

$$\{\ldots\}_{e\mu\to e\mu} = \left(\cos^2\theta/2 - \frac{q^2}{2m^2}\sin^2\theta/2\right)\delta\left(\nu + \frac{q^2}{2m}\right)$$

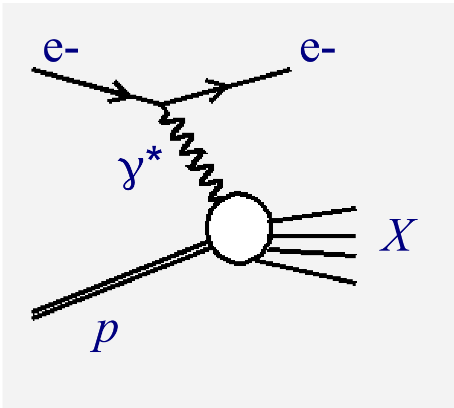
- ***ep* → *ep***

$$\{\ldots\}_{ep\to ep} = \left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\theta/2 + 2\tau G_M^2\sin^2\theta/2\right)\delta\left(\nu + \frac{q^2}{2M}\right)$$

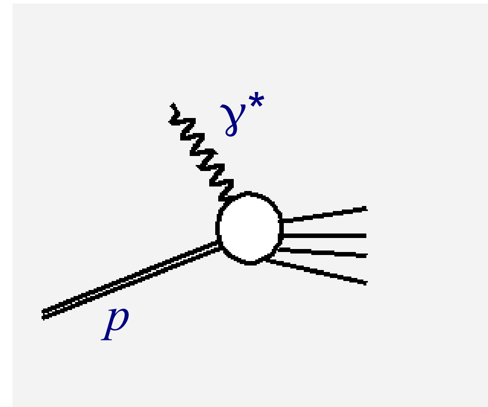
- DIS: ***ep* → *eX***

$$\{\ldots\}_{ep\to eX} = W_2(\nu,q^2)\cos^2\theta/2 + 2W_1(\nu,q^2)\sin^2\theta/2$$

Relationship between DIS and $\sigma_{tot}(\gamma^* p)$



what is important is:



- It is possible to show that for a **real photon (with transverse polarization)**:

$$\sigma_{tot}(\gamma\, p \to X) = \frac{4\pi^2\alpha}{K}\epsilon^{\mu *}\epsilon^{\nu} W_{\mu\nu}$$

where ϵ gives the photon polarization
while:

$$W^2 = (p+q)^2 = M^2 + 2MK$$

where K is the energy of the real photon.
The flux factor is $4MK$ with $K = \nu$

Relationship between DIS and $\sigma_{tot}(\gamma^* p)$

- For a **virtual photon** it is possible to demonstrate that:

$$\sigma_T(\gamma_T^* p \to X) = \frac{4\pi^2\alpha}{K} W_1(\nu, q^2) = \frac{4\pi^2\alpha}{\left(\nu + \frac{q^2}{2M}\right)} W_1(\nu, q^2)$$

transverse polarization

$$\sigma_L(\gamma_L^* p \to X) = \frac{4\pi^2\alpha}{K}\left(\left(1 - \frac{\nu^2}{q^2}\right) W_2(\nu, q^2) - W_1(\nu, q^2)\right)$$
$$= \frac{4\pi^2\alpha}{\left(\nu + \frac{q^2}{2M}\right)}\left(\left(1 - \frac{\nu^2}{q^2}\right) W_2(\nu, q^2) - W_1(\nu, q^2)\right)$$

longitudinal polarization

Crucial point:
For a virtual photon the cross section is not well defined. For a virtual photon ($q^2 \neq 0$) the flux is arbitrary. The conventional choice is to require K to continue to satisfy:

$$K = \frac{W^2 - M^2}{2M} = \nu + \frac{q^2}{2M}$$ Hand convention

in the laboratory frame.

Relationship between DIS and $\sigma_{tot}(\gamma^* p)$

- One can express $ep \to eX$ as function of $\sigma_{L,T}$:

$$\left(\frac{d\sigma}{dE' \, d\Omega}\right)_{lab} = \Gamma\left(\sigma_T + \epsilon\,\sigma_L\right)$$

Flux factor of virtual photons: $\Gamma = \frac{\alpha K}{2\pi^2 |q^2|} \frac{E}{E'} \frac{1}{1-\epsilon}$

the ratio of the flux for longitudinal to transverse photons $\epsilon = \left(1 - 2\frac{\nu^2 - q^2}{q^2} tg^2 \theta/2\right)^{-1}$

- If $q^2 \to 0$ that is $\gamma^* \to \gamma$ (real):

$$\sigma_T \to \sigma^{tot}(\gamma p)$$
$$\sigma_L \to 0$$

and also:

$$W_2 \to 0$$
$$\left(W_1 + \frac{\nu^2}{q^2} W_2\right) \to 0$$

Naive Quark-Parton Model: Bjorken Scaling

- **What happens if point-like, spin-½ objects reside inside the *p* are hitten with a small-wavelength ($\lambda = 1/\sqrt{-q^2} \ll 1$ fm) virtual photon beam?**
- the sign that there are structureless particles inside a complex system such as a *p*

$$\{\ldots\}_{ep\to eX} = W_2(\nu, q^2)\cos^2\theta/2 + 2W_1(\nu, q^2)\sin^2\theta/2$$

complex system

$$\downarrow$$

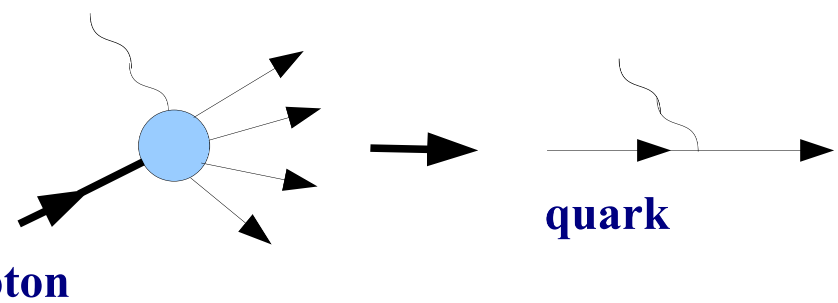
$$\{\ldots\}_{e\mu\to e\mu} = \left(\cos^2\theta/2 - \frac{q^2}{2\mathrm{m}^2}\sin^2\theta/2\right)\delta\left(\nu + \frac{q^2}{2\mathrm{m}}\right)$$

Dirac particles

$$\downarrow$$

$$2W_1^{point}(\nu, Q^2) = \frac{Q^2}{2m^2}\delta\left(\nu - \frac{Q^2}{2m}\right)$$

$$W_2^{point}(\nu, Q^2) = \delta\left(\nu - \frac{Q^2}{2m}\right)$$



where $Q^2 \equiv -q^2$

Naive Quark-Parton Model: Bjorken Scaling

- **That is for large Q^2, the *ep* inelastic scattering can be simple viewed as an elastic scattering on a free quark in the proton.**
  Using: $\delta(x/a) = a\delta(x)$:

$$2mW_1^{point}\left(\nu, Q^2\right) = \frac{Q^2}{2m\nu}\delta\left(1 - \frac{Q^2}{2m\nu}\right)$$
$$\nu W_2^{point}\left(\nu, Q^2\right) = \delta\left(1 - \frac{Q^2}{2m\nu}\right)$$

Pointlike functions have the property to depend only on the ratio $Q^2/2m\nu$ but not on Q^2 and ν independently.

- What does it happen in the case *ep* → *ep*. If we put for simplicity $\xi = 0$ $G_E = G_M \equiv G$

$$W_1^{elastic} = \frac{Q^2}{4M^2} G(Q^2)\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$
$$W_2^{elastic} = G(Q^2)\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

These structure functions contain the factor $G(Q^2)$. It is not possible to rearrange them as function of a single dimensionless variable. A mass scale is explicitly present.

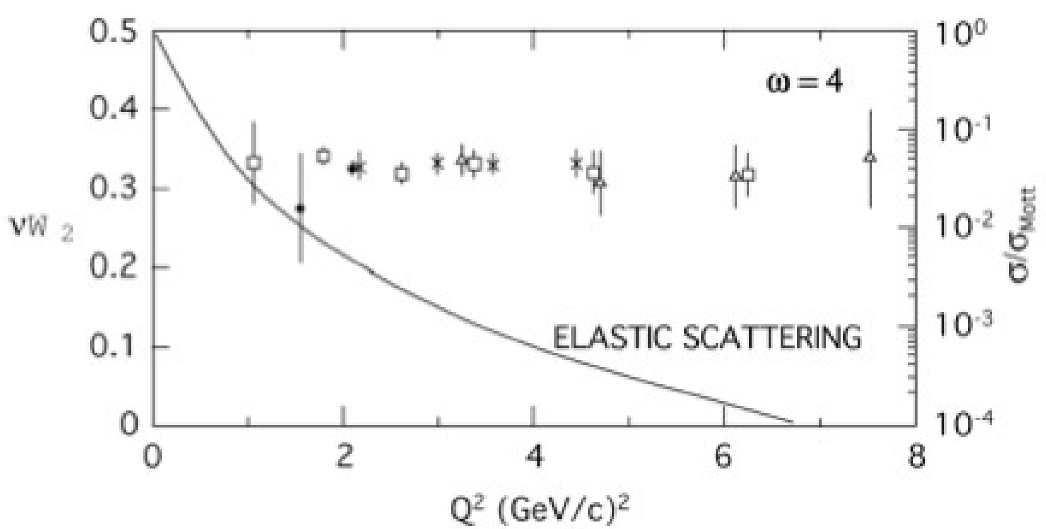
Naive Quark-Parton Model: Bjorken Scaling

- Summarizing:

$$MW_1(\nu, Q^2) \xrightarrow[\text{large } Q^2]{} F_1(\omega)$$
$$\nu W_2(\nu, Q^2) \xrightarrow[\text{large } Q^2]{} F_2(\omega)$$

$$\omega = \frac{2q\cdot p}{Q^2} = \frac{2M\nu}{Q^2}$$

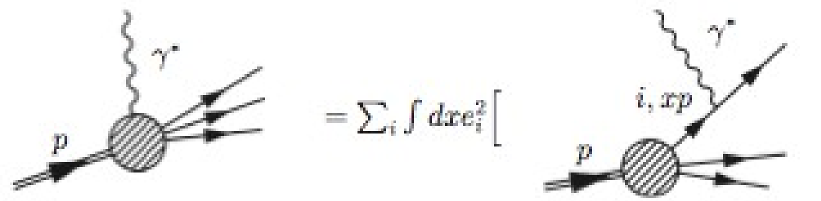
- **The presence of free *q* is signalled by the fact that the inelastic structure functions do not depend on Q^2 at a given value of ω.** This is equivalent to the onset of $1/\sin^4 \theta/2$ behaviour for large momentum transfers in the Rutherford experiment.



Are these pointlike particles (called **partons**) the **quarks** of the hadrons spectroscopy?

Partons and Bjorken Scaling

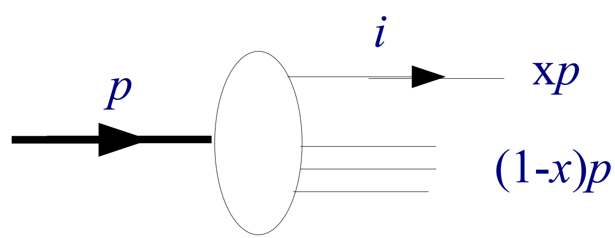
- The process $\gamma^* p \rightarrow X$ can be seen as:



The partons can transport a variable fraction x of the 4-momentum of the p.

- One introduces the **parton momentum distribution**:

$$f_i(x) = \frac{dP_i}{dx} =$$



describes the probability that the struck parton i carries a fraction x of the proton's momentum p

Moreover (i sums over all partons, not just the charged ones which interact with the photon):

$$\sum_i \int dx\, x\, f_i(x) = 1$$

Partons and Bjorken Scaling

- **Kinematics**:

	Proton	**Parton**
Energy	E	xE
Momentum	p_L	xp_L
	$p_T = 0$	$p_T = 0$
Mass	M	$m = (x^2E^2 - x^2p^2_L)^{1/2} = xM$

Partons and Bjorken Scaling

- For an e- hitting a **parton** with momentum x and unit charge, the dimensionless structure functions are:

$$F_1(\omega)=\frac{Q^2}{4m\nu x}\delta\left(1-\frac{Q^2}{2m\nu}\right)=\frac{1}{2x^2\omega}\delta\left(1-\frac{1}{x\omega}\right)=\frac{1}{2x\omega}\delta\left(x-\frac{1}{\omega}\right)$$

$$F_2(\omega)=\delta\left(1-\frac{Q^2}{2m\nu}\right)=\delta\left(1-\frac{Q^2}{2xM\nu}\right)=\delta\left(1-\frac{1}{x\omega}\right)=\delta\left(\frac{1}{x}\left(x-\frac{1}{\omega}\right)\right)=x\delta\left(x-\frac{1}{\omega}\right)$$

- for a **proton**:

$$F_2(\omega)=\sum_i\int dx\, e_i^2 f_i(x)\, x\,\delta\left(x-\frac{1}{\omega}\right)=\sum_i e_i^2 f_i\left(\frac{1}{\omega}\right)\frac{1}{\omega}$$

$$F_1(\omega)=\sum_i\int dx\, e_i^2 f_i(x)\frac{1}{2x\omega}\delta\left(x-\frac{1}{\omega}\right)=\frac{1}{2}\sum_i e_i^2 f_i\left(\frac{1}{\omega}\right)=\frac{\omega}{2}F_2(\omega)$$

- It is conventional to redefine $F_{1,2}(\omega)$ as $F_{1,2}(x)$:

$$\nu W_2(\nu,Q^2)\quad\rightarrow\quad F_2(x)=\sum_i e_i^2\, x f_i(x)$$
$$W_1(\nu,Q^2)\quad\rightarrow\quad F_1(x)=\frac{1}{2x}F_2(x)$$

$$x=\frac{1}{\omega}=\frac{Q^2}{2M\nu}$$

F_1 e F_2 are functions of only one variable, namely, x. They are independent of Q^2 at fixed x. We say they satisfy **Bjorken scaling**.

Partons and Bjorken Scaling

$$\nu W_2(\nu, Q^2) \rightarrow F_2(x) = \sum_i e_i^2 x f_i(x)$$
$$W_1(\nu, Q^2) \rightarrow F_1(x) = \frac{1}{2x} F_2(x)$$

$$x = \frac{1}{\omega} = \frac{Q^2}{2M\nu}$$

- F_1 e F_2 are functions of only one variable, namely, x. They are independent of Q^2 at fixed x. We say they satisfy **Bjorken scaling**.
- The momentum fraction transported by the parton is found to be identical to the (dimensionless) kinematic variable x of the virtual photon. The virtual photon must have just the right value of the variable x to be absorbed by a parton with momentum fraction x.
  It is the function $\delta(x - 1/\omega)$ that equates these two distinct physical variables.

Partons and Bjorken Scaling

- Kinematics observations (see formulas at pag. 23):
  The calculations are computed in a Lorentz frame where:

$$|\vec{p}| \gg m\,,M$$

in this frame the proton is moving with infinite momentum: here kinematics + $F_{1,2}(x)$ become exact.

- In this particular reference frame (**infinite momentum frame**) relativistic time dilation slows down the rate at which partons interact with one another. During the short time of the interaction the quark is essentially a free particle, not interacting with its friends in the proton.

- At the end, the struck colored parton has to recombine with the noninteracting spectator partons to form the colorless hadrons into which the proton breaks up. This happens with probability 1. The asymptotic freedom is used to calculate the cross section, the confinement process does not affect the result.

- **This is valid when Q^2 and W are both large.**

Partons and Bjorken Scaling

- Another manner to show the **cross section** in an **invariant manner:**

$$dE' \, d\Omega = \frac{\pi}{E E'} d Q^2 d\nu = \frac{2 M E}{E'} \pi \, y \, dy \, dx$$

knowing that:

$$x = \frac{Q^2}{2 M \nu}; \qquad y = \frac{p \cdot q}{p \cdot k} = \left(\frac{\nu}{E} \right)_{lab}$$

one obtains (where $\nu_{max} = E$ in the lab. frame)

$$M \nu_{max} \frac{d\sigma}{dx\, dy} = \frac{2\pi\alpha^2}{x^2 y^2} \left\{ x y^2 F_1 + \left[(1-y) - \frac{M x y}{2 \nu_{max}} \right] F_2 \right\}$$

from this one can write:

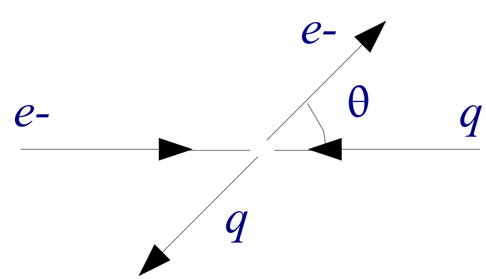
eq with the same helicity

eq with opposite helicity

$$\left(\frac{d\sigma}{dx\, dy} \right)_{ep \to eX} = \frac{2\pi\alpha^2}{Q^4} s \left[1 + (1-y)^2 \right] \sum_i e_i^2 \, x f_i(x)$$

moreover

$$1 - y = \frac{p \cdot k'}{p \cdot k} \approx \frac{1}{2} (1 + \cos\theta)$$



Partons and Bjorken Scaling

- **Callan-Gross relationship:**

$$\boxed{2x F_1(x) = F_2(x)} \longrightarrow \text{From the fact that } q \text{ have spin 1/2}$$

- **Meaning of the Callan-Gross relationship**
  - real photons have $Q^2 = 0$ and can exist only with transverse helicity ($\lambda = \pm 1$);
  - virtual photons have $Q^2 \neq 0$ and can exist also with longitudinal helicity (or scalar) ($\lambda = 0$). Virtual photons behave as spin 1 particles with mass different from zero.
- The cross section for virtual photons is:

$$\sigma_T(\gamma_T^* p \rightarrow X) = \frac{4\pi^2\alpha}{K} W_1(\nu, q^2)$$

$$\sigma_L(\gamma_L^* p \rightarrow X) = \frac{4\pi^2\alpha}{K}\left(\left(1-\frac{\nu^2}{q^2}\right)W_2(\nu,q^2) - W_1(\nu,q^2)\right)$$

- Due to the fact that the cross sections must be positive or null, one has:

$$W_1 \geq 0$$

$$\left(1-\frac{\nu^2}{q^2}\right)W_2 - W_1 \geq 0$$

Partons and Bjorken Scaling

- At the limit of the Bjorken scaling $Q^2 \rightarrow \infty$, $\nu \rightarrow \infty$ and $x = Q^2/2M\nu$ is fixed:

$$\sigma_T \rightarrow \frac{4\pi\alpha^2}{KM} F_1(x)$$

$$\sigma_L \rightarrow \frac{4\pi\alpha^2}{KM} \frac{1}{2x} \left[F_2(x) - 2xF_1(x) \right]$$

- **Breit Frame**

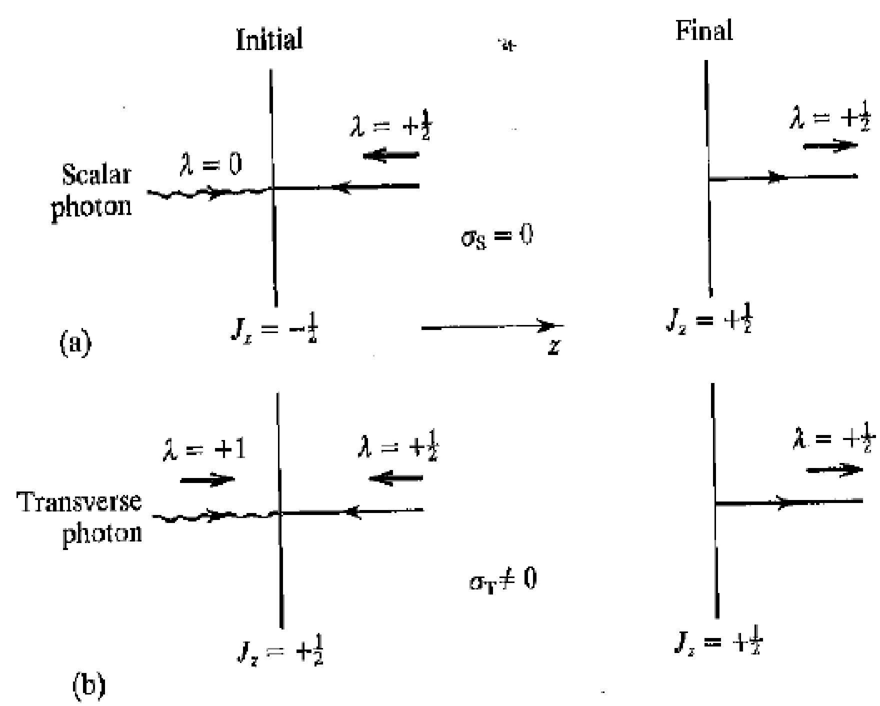


Figure 12.17
Absorption of (a) scalar and (b) transverse photons by helicity conserving partons viewed in the Breit frame of reference.

Partons and Bjorken Scaling

- From the analysis of the figures:

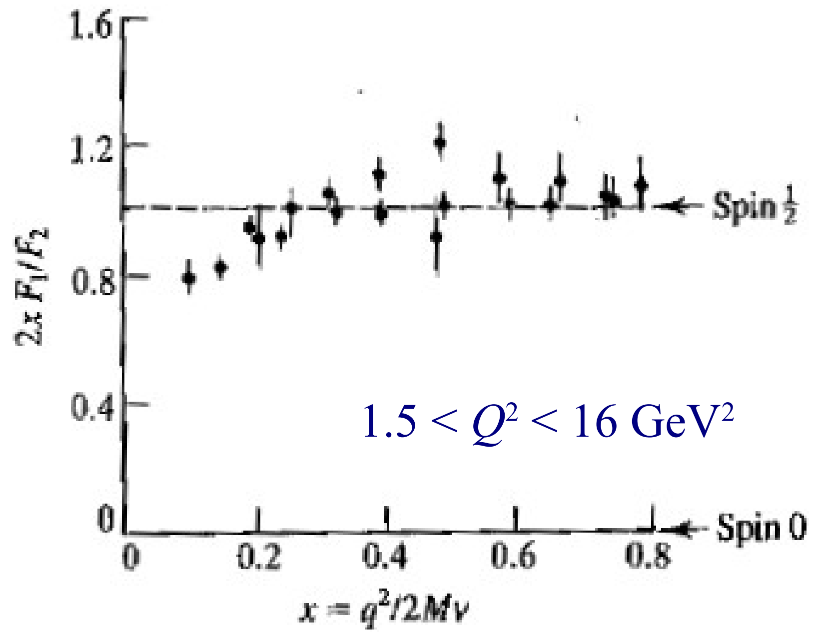
  - If the q have spin ½ it is not possible to absorb a scalar photon: then $\sigma_L \rightarrow 0$ if the Callan-Gross relationship is true

  - If the q have spin ½ they are able to absorb transverse photons: $\sigma_T \neq 0$

  - Thus, if the q have spin ½, $\sigma_L/\sigma_T \rightarrow 0$ and the Callan Gross relationship should be valid

  - If the q have spin 0, they can't absorb transverse photons and $\sigma_T/\sigma_L \rightarrow 0$

- Experimental results:



The quarks within the proton

- **Proton** (excluding charm and heavier contributions):

$$\frac{1}{x}F_2^{ep}(x)=\sum_i e_i^2 f_i(x)=\left(\frac{2}{3}\right)^2\left[u^p(x)+\bar{u}^p(x)\right]+\left(\frac{1}{3}\right)^2\left[d^p(x)+\bar{d}^p(x)\right]+\left(\frac{1}{3}\right)^2\left[s^p(x)+\bar{s}^p(x)\right]$$

- **Neutron** (from the scattering ***ed → eX***):

$$\frac{1}{x}F_2^{en}(x)=\sum_i e_i^2 f_i(x)=\left(\frac{2}{3}\right)^2\left[u^n(x)+\bar{u}^n(x)\right]+\left(\frac{1}{3}\right)^2\left[d^n(x)+\bar{d}^n(x)\right]+\left(\frac{1}{3}\right)^2\left[s^n(x)+\bar{s}^n(x)\right]$$

- p and n are members of the same isospin doublet: f_i^n are correlated with f_i^p

$$u^p(x)=d^n(x)\equiv u(x)$$
$$d^p(x)=u^n(x)\equiv d(x)$$
$$s^p(x)=s^n(x)\equiv s(x)$$

The quarks within the proton

- Further constraints come from the fact that $p \equiv u_v u_v d_v$ (valence quarks) and $n \equiv d_v d_v u_v$ (valence quarks) then there are many pairs $u_s \bar{u}_s$, $d_s \bar{d}_s$, ... (see quarks):

$$u_s = \bar{u}_s = d_s = \bar{d}_s = s_s = \bar{s}_s = S(x)$$
$$u = u_v + u_s$$
$$d = d_v + d_s$$

- **Sum rules**:

$$\int_0^1 [u - \bar{u}] dx = 2$$
$$\int_0^1 [d - \bar{d}] dx = 1$$
$$\int_0^1 [s - \bar{s}] dx = 0$$

in this manner one recovers the quantum numbers of the proton: charge 1, baryon number 1, strangeness 0.

The quarks within the proton

- $$\frac{1}{x}F_2^{ep}=\frac{1}{9}\left[4u_v+d_v\right]+\frac{4}{3}S \qquad (*)$$
  $$\frac{1}{x}F_2^{en}=\frac{1}{9}\left[u_v+4d_v\right]+\frac{4}{3}S$$

- $S(x)$ comes from the splitting in quark-antiquark pairs of the gluons. $S(x)$ has a bremsstrahlung spectrum at low x: the number of the sea quarks grows as ln (x) for $x \rightarrow 0$:

$$f_i(x) \xrightarrow[x\to 0]{} \frac{1}{x}$$

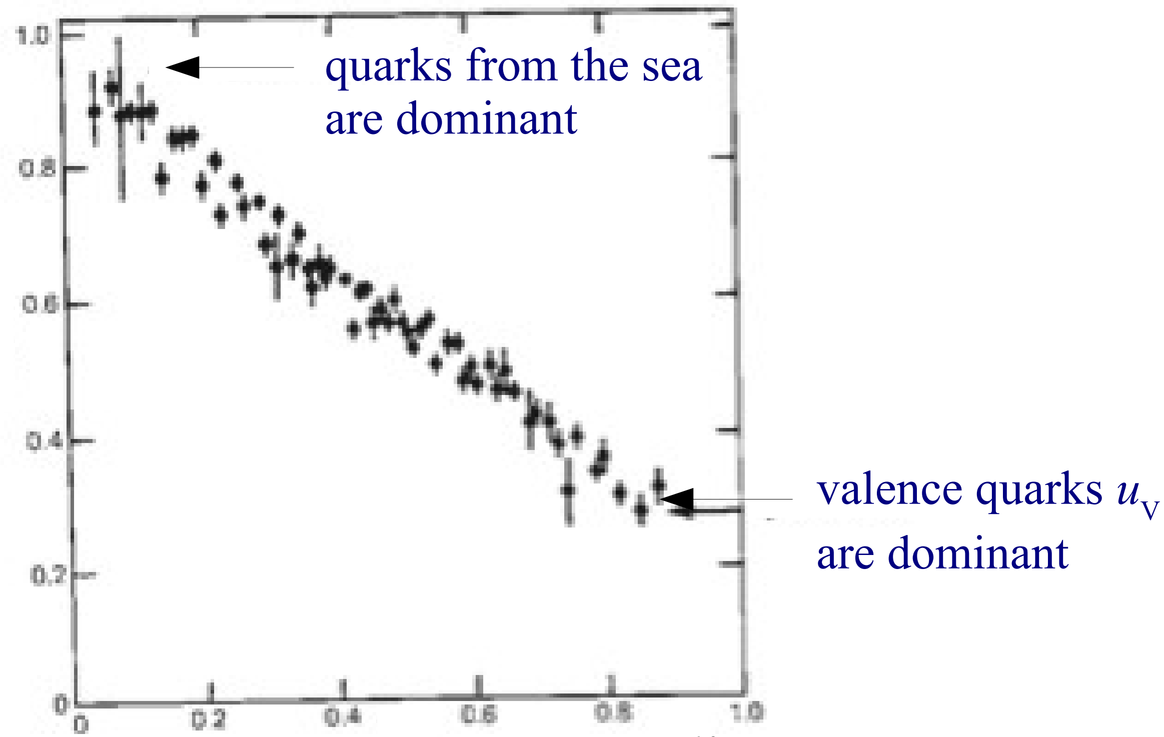
From (*) one has:

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow[x\to 0]{} 1 \qquad\qquad \frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow[x\to 1]{} \frac{u_v+4d_v}{4u_v+d_v}$$

The quarks within the proton

-

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)}$$



x

The quarks within the proton

- The **counting rules** (*) give, in the limit in which a single parton i carries all the momentum of the proton, ($n_s \equiv$ number of spectator valence quarks):

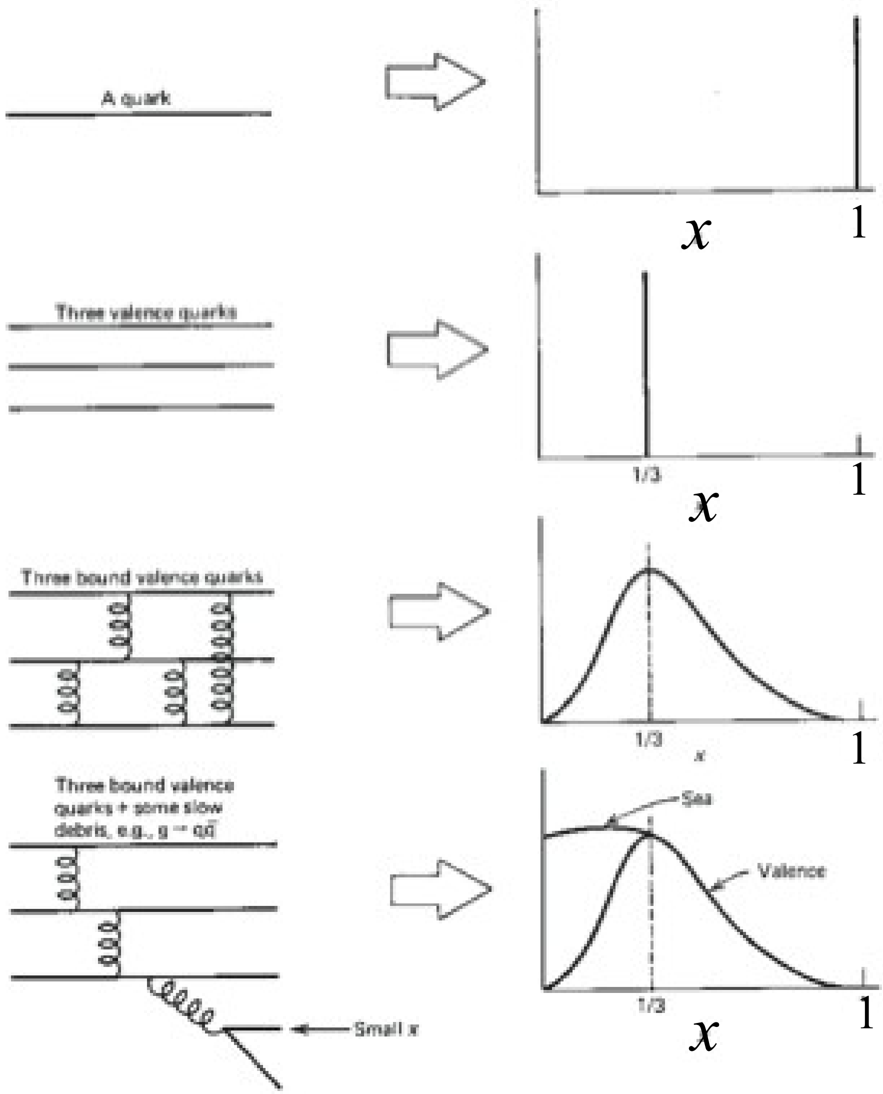
$$\underset{x \to 1}{f_i(x) \to} (1-x)^{2n_s - 1}$$

For a valence quark in a nucleon: $q_i = (1-x)^3$
For a sea quark in a nucleon: $q_i = (1-x)^7$
For a gluon in a nucleon: $q_i = (1-x)^5$
For a valence quark in a meson: $q_i = (1-x)$
For a sea quark in a meson: $q_i = (1-x)^5$
For a gluon in a meson: $q_i = (1-x)^3$

(*) **counting rules** $\equiv$ one counts the number of quarks not participating directly in the interation. More spectators subdividing the initial moment, less it is the possibility to produce a parton with a high momentum fraction.

The quarks within the proton

- If the proton is … F_2^{ep} is



the momentum distribution is enlarged due to the interactions

situation at a certain Q^2

The quarks within the proton

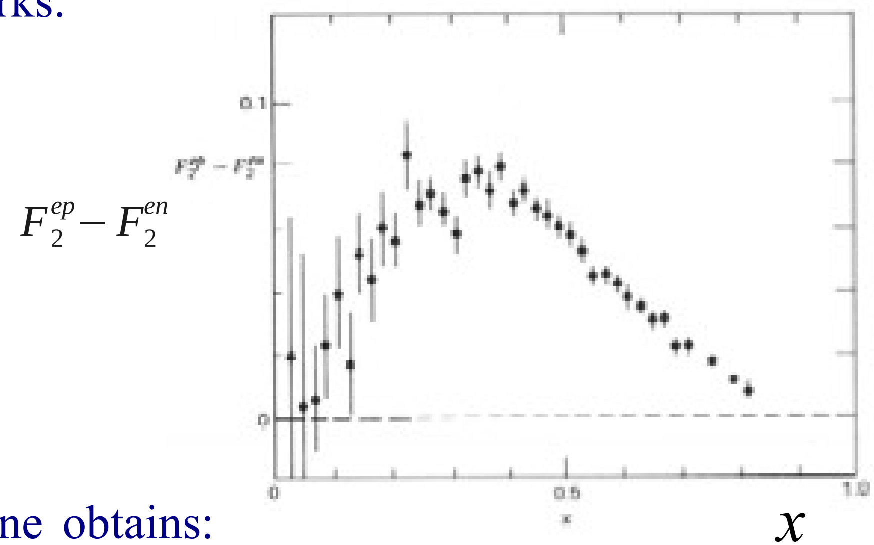
- By subtracting, we can observe the valence quarks:

$$\frac{1}{x}\left[F_2^{ep}-F_2^{en}\right] = \frac{1}{3}\left[u_v(x)-d_v(x)\right]$$

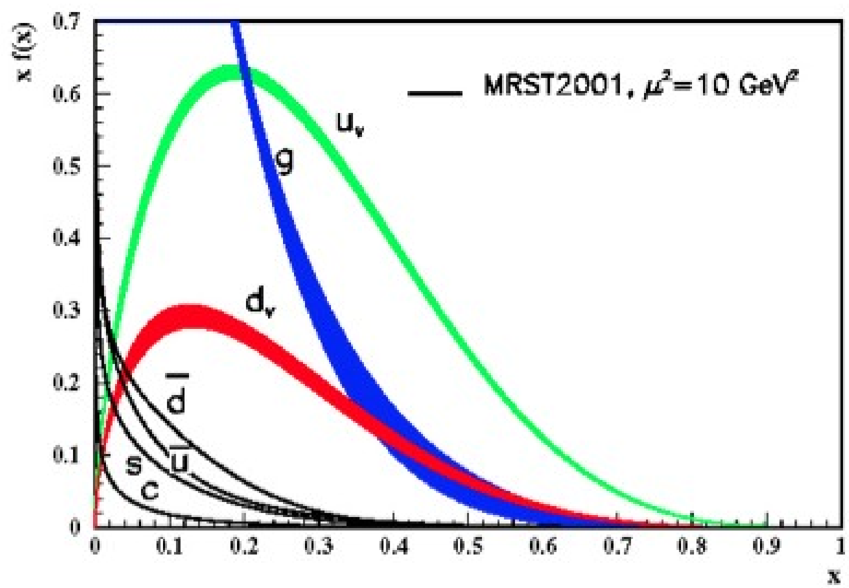
starting from:

$$\frac{1}{x}F_2^{ep}=\frac{1}{9}\left[4u_v+d_v\right]+\frac{4}{3}S$$

$$\frac{1}{x}F_2^{en}=\frac{1}{9}\left[u_v+4d_v\right]+\frac{4}{3}S$$



- Parameterizing all the data available for $F_2^{ep,en}$ one obtains:



Gluons

- $$\int_0^1 dx\,(xp)\left[u+\bar{u}+d+\bar{d}+s+\bar{s}\right] = p-p_g$$
  $$\int_0^1 dx\,x\left[u+\bar{u}+d+\bar{d}+s+\bar{s}\right] = 1-\epsilon_g$$

$\epsilon_g = p_g/p$ momentum fraction carried by the gluons. The gluons cannot be detected by the γ (the gluons do not have electric charge).

$$\int dx\,F_2^{ep}(x)=\frac{4}{9}\epsilon_u+\frac{1}{9}\epsilon_d=0.18$$

$$\int dx\,F_2^{en}(x)=\frac{1}{9}\epsilon_u+\frac{4}{9}\epsilon_d=0.12$$

$$\epsilon_u\equiv\int_0^1 dx\,x\,(u+\bar{u})$$

(after neglecting the strange quarks which carry a small fraction of the nucleon's momentum)

$$\epsilon_g = 1 - \epsilon_u - \epsilon_d$$

solving:

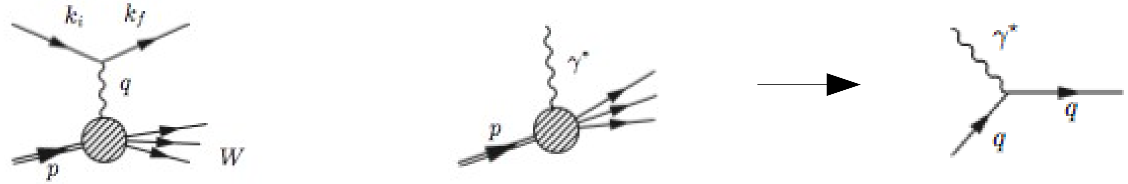
- $\epsilon_u = 0.36$ $\epsilon_d = 0.18$ $\boxed{\epsilon_g = 0.46}$ **The gluon carry ~50% of the proton momentum**

Gluons

From the analysis of DIS data:

- **Bjorken scaling** → presence of point-like Dirac particles inside hadrons. They are called **partons**
- From the study of their quantum numbers: **partons ≡ quarks** of the hadron spectroscopy
- There are **neutral partons → gluons** of the QCD

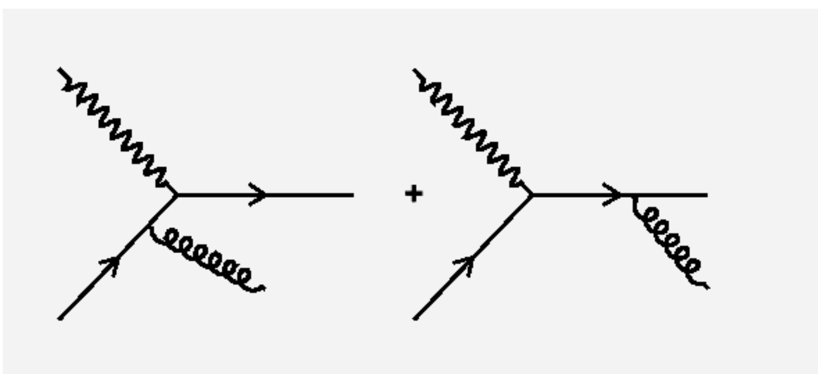
Improved Parton Model

- 

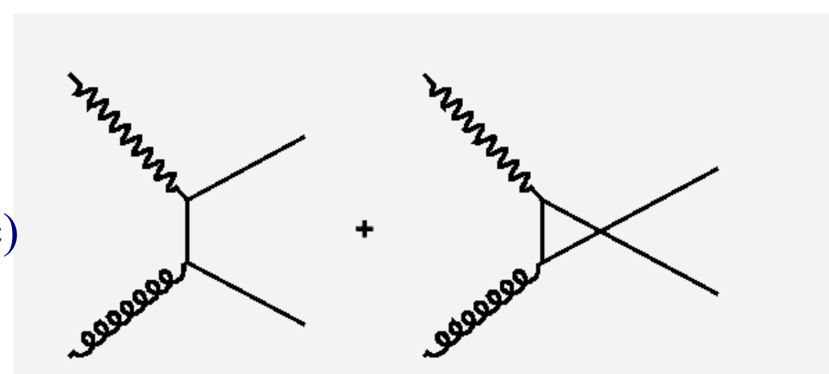
(a)

ep → *eX*
in the parton model
No gluons

- Beyond the graph (a) of α order, there are also:

(b) 

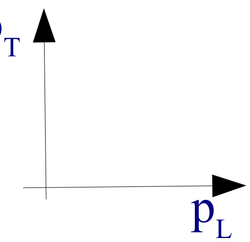
process: $\gamma^* q \rightarrow qg$, contribution of $O(\alpha\alpha_s)$
QCD Compton scattering

(c) 

process: $\gamma^* g \rightarrow q\bar{q}$, contribution of $O(\alpha\alpha_s)$
Boson Gluon Fusion

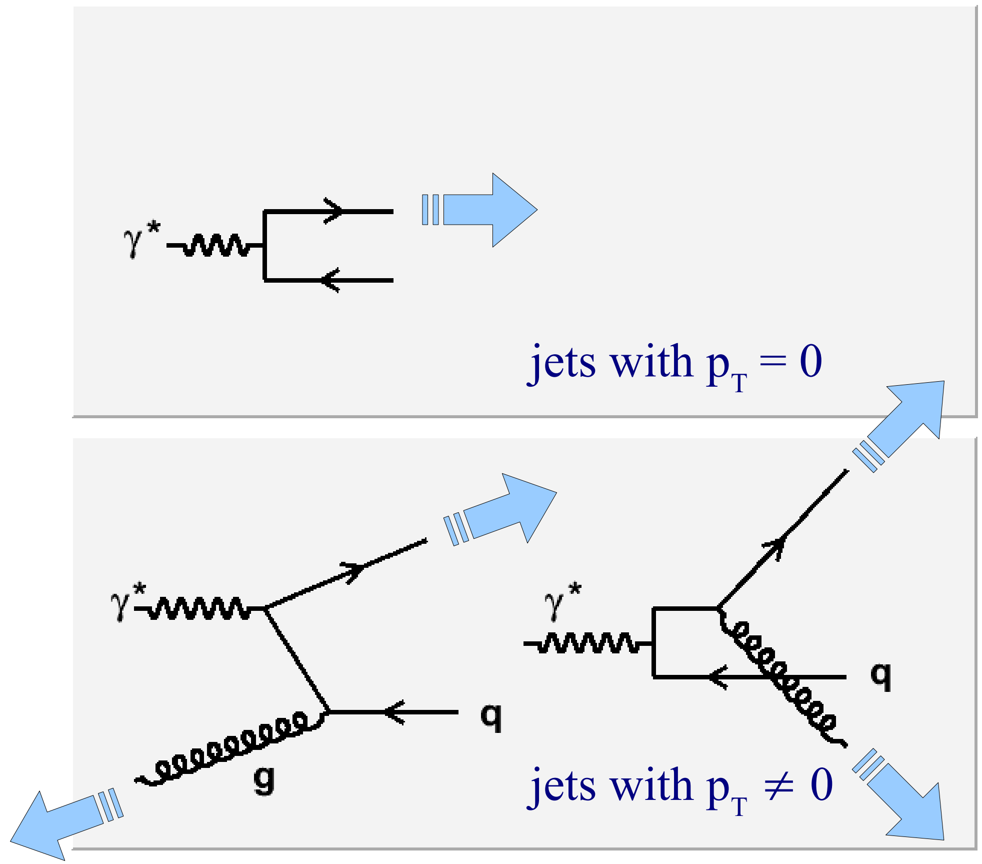
Improved Parton Model

- The inclusion of (b) e (c) in the calculation of the DIS process has 2 experimentally observable consequences:
  - Scaling violation of the structure functions;
  - the outgoing *q* (the hadronic jet) will be no more collinear with the γ^*:



process $\gamma^* q \rightarrow q$

Improved Parton Model



Improved Parton Model

- In the past we have written for $\gamma^*p \to X$:

$$\sigma_T = \frac{4\pi^2\alpha}{K} W_1 = \frac{4\pi^2\alpha}{KM} M W_1 = \frac{4\pi^2\alpha}{2KM} 2MW_1 = 2\sigma_0 F_1 \xrightarrow{\text{in the DIS limit}} \boxed{2F_1 = \frac{\sigma_T}{\sigma_0}}$$

$$\sigma_L = \frac{4\pi^2\alpha}{K}\left[\left(1-\frac{\nu^2}{q^2}\right)W_2 - W_1\right] = \frac{4\pi^2\alpha}{KM}\left[M\left(1+\frac{\nu^2}{Q^2}\right)W_2 - MW_1\right]$$

$$= \frac{4\pi^2\alpha}{2KM}\left[2M\left(1+\frac{\nu^2}{Q^2}\right)W_2 - 2F_1\right] = \sigma_0\left[\frac{2M}{\nu}\left(1+\frac{\nu^2}{Q^2}\right)\nu W_2 - 2F_1\right]$$

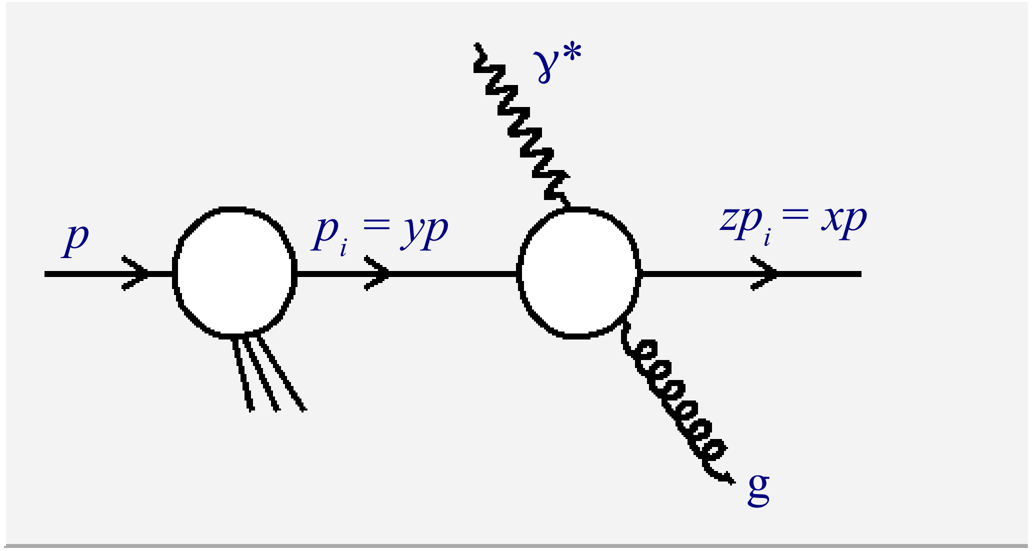
$$\boxed{\sigma_0 = \frac{4\pi^2\alpha}{2KM} \simeq \frac{4\pi^2\alpha}{s}}$$

$$= \sigma_0\left[\frac{2M}{\nu}F_2 + \frac{1}{x}F_2 - 2F_1\right] \simeq \sigma_0\left[\frac{1}{x}F_2 - 2F_1\right]$$

$$\frac{\sigma_L}{\sigma_0} = \frac{F_2}{x} - \frac{\sigma_T}{\sigma_0} \xrightarrow{\text{in the DIS limit}} \boxed{\frac{F_2}{x} = \frac{\sigma_T+\sigma_L}{\sigma_0}}$$

Improved Parton Model

- The two formulas are valid for $\gamma^* p$ now we have to translate them for γ^*-parton.
- Here the problem for the case $\gamma^* q \rightarrow qg$:



γ^*- proton		γ^*- parton
p	$\longrightarrow$	$p_i = yp$
$x = \dfrac{Q^2}{2\, p \cdot q}$	$\longrightarrow$	$z = \dfrac{Q^2}{2\, p_i \cdot q} = \dfrac{x}{y}$

We have relied extensively on the fact that both collinear frames move with infinite momentum.

Improved Parton Model

- We can now write the decomposition of F_1 and F_2 respect to the γ*-parton cross sections:

$$2F_1 = \left(\frac{\sigma_T(x,Q^2)}{\sigma_0}\right)_{\gamma^* p} = \sum_i \int_0^1 dz \int_0^1 dy\, f_i(y)\,\delta(x-zy)\left(\frac{\hat{\sigma}_T(z,Q^2)}{\hat{\sigma}_0}\right)_{\gamma^* i}$$

$$= \sum_i \int_x^1 \frac{dy}{y} f_i(y)\left(\frac{\hat{\sigma}_T(x/y,Q^2)}{\hat{\sigma}_0}\right)_{\gamma^* i}$$

$$\frac{F_2}{x} = \left(\frac{\sigma_T(x,Q^2)+\sigma_L(x,Q^2)}{\sigma_0}\right)_{\gamma^* p} = \sum_i \int_0^1 dz \int_0^1 dy\, f_i(y)\,\delta(x-zy)\left(\frac{\hat{\sigma}_T(z,Q^2)+\hat{\sigma}_L(z,Q^2)}{\hat{\sigma}_0}\right)_{\gamma^* i}$$

$$= \sum_i \int_x^1 \frac{dy}{y} f_i(y)\left(\frac{\hat{\sigma}_T(x/y,Q^2)+\hat{\sigma}_L(x/y,Q^2)}{\hat{\sigma}_0}\right)_{\gamma^* i}$$

Improved Parton Model

- **If there are no gluons, there is only the process $\gamma^* q \to q$.**
  Neglecting the mass of the outgoing quark one has $(q + p_i)^2 = 0$, and then:

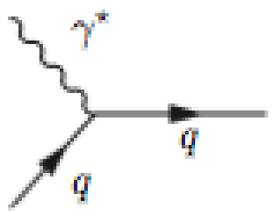
$$z = \frac{Q^2}{2\, p_i q} = 1$$

$$\frac{\hat{\sigma}_T(z, Q^2)}{\hat{\sigma}_0} = e_i^2\, \delta(1-z)$$

$$\hat{\sigma}_L(z, Q^2) = 0$$

it can be demonstrated

$$\frac{\hat{\sigma}_T(z, Q^2) + \hat{\sigma}_L(z, Q^2)}{\hat{\sigma}_0} = e_i^2\, \delta(1-z)$$



- Putting together the previous results, one obtains:

$$\frac{F_2}{x} = \sum_i e_i^2 \int_x^1 \frac{dy}{y} f_i(y)\, \delta\left(1 - \frac{x}{y}\right) = \sum_i e_i^2 f_i(x)$$

$$F_1 = \frac{1}{2} \sum_i e_i^2 f_i(x)$$

Main result of the parton model
The adopted formalism is consistent

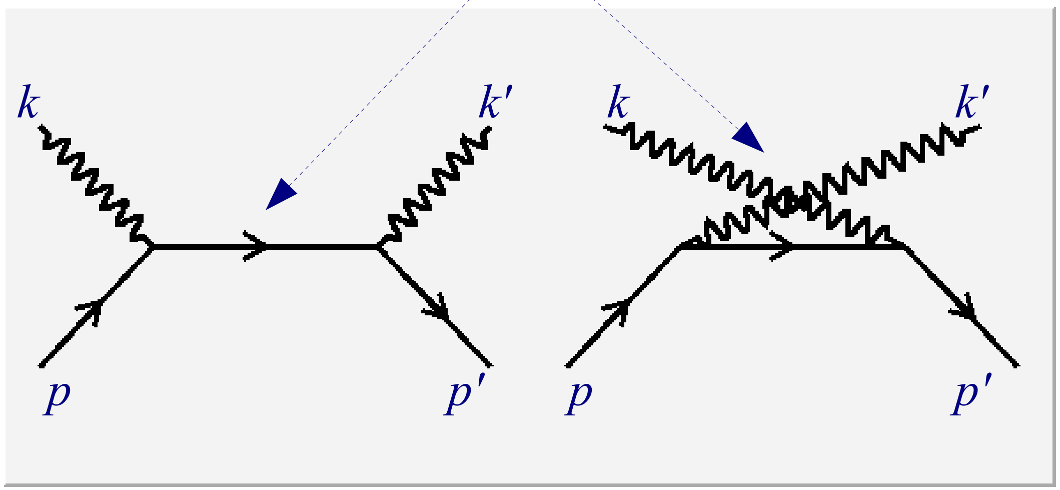
Improved Parton Model

- **We include** $\gamma^* q \to qg$
  $\gamma^* q \to qg$ **similar to the Compton**: $\gamma^* e \to \gamma e$:

$$\overline{|M|^2} = 32\pi^2\alpha^2\left(-\frac{u}{s}-\frac{s}{u}+\frac{2tQ^2}{su}\right)$$

Term due to the fact that the γ^* virtuality is: $Q^2 \neq 0$

γ^*-polar. sum:
$\sum \epsilon^*_\mu \epsilon_\nu = -g_{\mu\nu}$

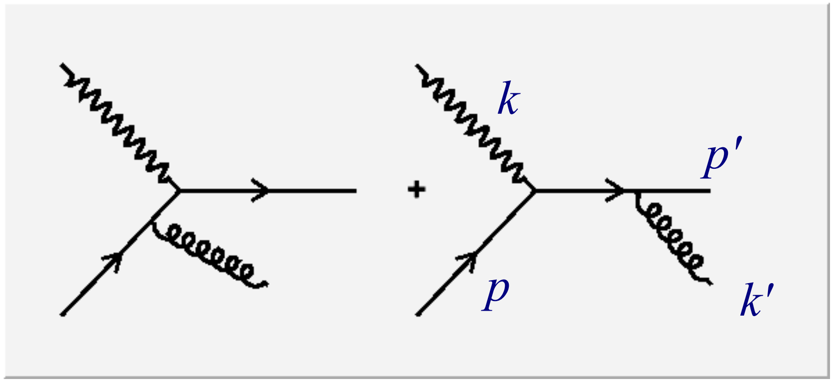


$$\gamma(k)+e(p)\to\gamma(k')+e(p')$$
$$s = (k+p)^2$$
$$t = (k-k')^2$$
$$u = (k-p')^2$$

Improved Parton Model

- $\gamma^* q \to qg$:

$$\overline{|M|^2} = 32\pi^2 (e_i^2 \alpha\, \alpha_s) \frac{4}{3} \left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\,\hat{t}} \right)$$



$\alpha^2 \to e^2_{\,i}\, \alpha\alpha_s$

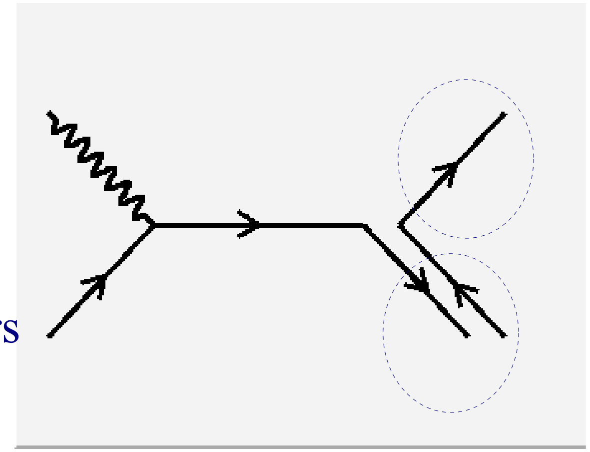
color factor : 4/3 ≡ summation/averaging over final/initial colors

$u \leftrightarrow t$ on account of the different ordering of the outgoing particles: $\gamma \to q$; $e \to g$

$$\hat{t} = (k - p')^2 \leftrightarrow u$$
$$\hat{u} = (k - k')^2 \leftrightarrow t$$

- Counting of the **color lines**:

3 colors



3 colors × 3 quarks lines →
3×3 = 9 possibilities of outgoing colors
-1 no singlet
8

4/3 = 8/3 ×1/2

from the α_s definition

average of the initial colors

Improved Parton Model

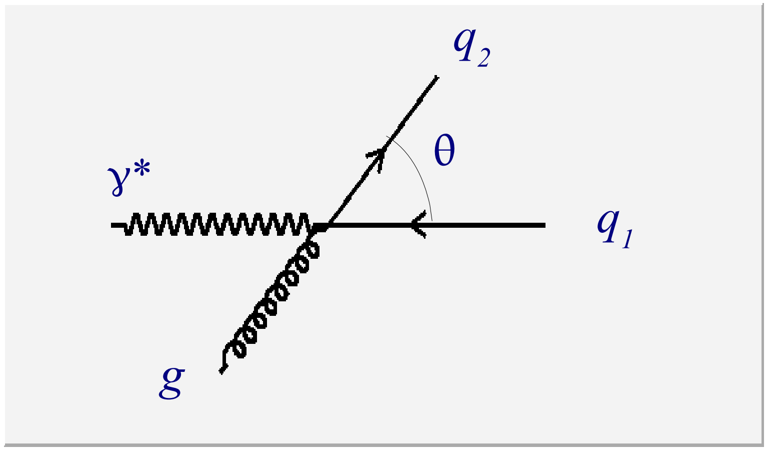
- Most simple QCD diagrams ~ QED diagrams with the substitution in the cross section of:

$$\alpha^n \quad \rightarrow \quad C_F \alpha_s^n$$

$n \equiv$ number of *qg* or *gg* vertices
C_F = color factor (summing and averaging the colors, in much the same way we do the spin)

Improved Parton Model

- **Kinematics** of : $\gamma^* q_1 \to q_2 g$



$$q = (q_0, 0, 0, k); \qquad q_1 = (k, 0, 0, -k)$$
$$q_2 = (k', k'\sin\theta, 0, k'\cos\theta); \quad g = (k', -k'\sin\theta, 0, -k'\cos\theta)$$
$$\hat{s} = (q+q_1)^2 = (q_2+g)^2 = 2k^2 + 2kq_0 - Q^2 = 4k'^2$$
$$\hat{t} = (q-q_2)^2 = (g-q_1)^2 = -Q^2 - 2k'q_0 + 2kk'\cos\theta$$
$$= -2kk'(1-\cos\theta)$$
$$\hat{u} = (q_1-q_2)^2 = -2kk'(1+\cos\theta)$$

for a virtual photon: $q_0^2 = k^2 - Q^2$

a useful result: $4kk' = -\hat{t} - \hat{u} = \hat{s} + Q^2$

- The important quantity is the **trasverse momentum of the outgoing quark**:

$$p_T^2 = (k'\sin\theta)^2 = \frac{\hat{s}\hat{t}\hat{u}}{(\hat{s}+Q^2)^2} \rightarrow \frac{\hat{s}(-\hat{t})}{\hat{s}+Q^2}$$

in the case of scattering at small angle: $-\hat{t} \ll \hat{s}$

- For small scattering angle ($\cos\theta \simeq 1$):

$$d\Omega = \frac{4\pi}{\hat{s}} d p_T^2$$

Improved Parton Model

- For large $\hat{s}$ (high energy), $\sigma(\gamma^* q \to qg)$ has a peak when $-\hat{t} \to 0$: in this case the q exchange in the t channel

- One can approximate the cross section with its forward peak:

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{1}{64\pi^2 \hat{s}} \frac{p_f}{p_i} \overline{|M|^2} = \frac{1}{64\pi^2 \hat{s}} \overline{|M|^2} \to \frac{d\hat{\sigma}}{d p_T^2} \simeq \frac{1}{16\pi \hat{s}^2} \overline{|M|^2}$$

- substituting the matrix element in the preceding formula:

$$\frac{d\hat{\sigma}}{d p_T^2} \simeq \frac{1}{16\pi \hat{s}^2} 32\pi^2 (e_i^2 \alpha \alpha_s) \frac{4}{3} \left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}} \right)$$

$$= \frac{8\pi e_i^2 \alpha \alpha_s}{3\hat{s}^2} \left(\frac{1}{-\hat{t}} \right) \left[\hat{s} + \frac{2(\hat{s}+Q^2)Q^2}{\hat{s}} \right]$$

$$4kk' = -\hat{t} - \hat{u} = \hat{s} + Q^2$$

having always made the approximation good for the regime: $-\hat{t} \ll \hat{s}$

Improved Parton Model

- then from:

$$z \equiv \frac{Q^2}{2\,p_i q} = \frac{Q^2}{(p_i+q)^2-q^2} = \frac{Q^2}{\hat{s}+Q^2}$$

- and from:

$$p_T^2 = \frac{\hat{s}(-\hat{t})}{\hat{s}+Q^2} \qquad \text{for} \qquad -\hat{t} \ll \hat{s}$$

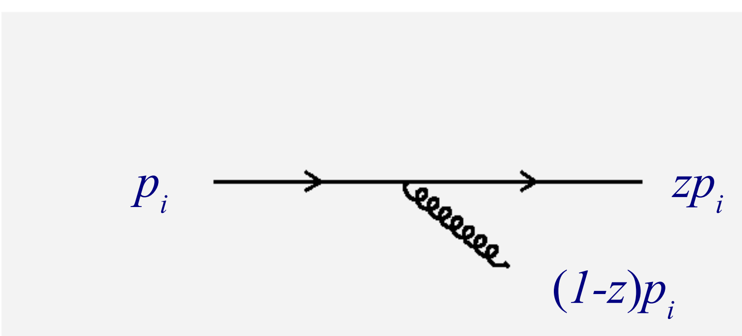
- one can write:

$$\frac{d\,\hat{\sigma}}{d\,p_T^2} \simeq e_i^2 \hat{\sigma}_0 \frac{1}{p_T^2}\frac{\alpha_s}{2\pi} P_{qq}(z)$$

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$

with $\hat{\sigma}_0 = \dfrac{4\pi^2\alpha}{\hat{s}}$

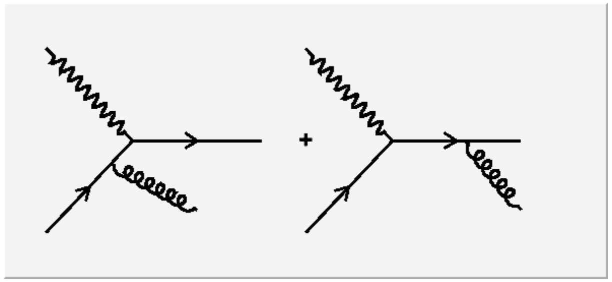
Probability of a *q* emitting a *g*, and so becoming a *q* with momentum reduced by a fraction *z*



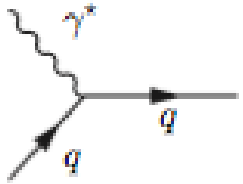
$P_{qq}(z)$ diverges for $z \to 1$: emission of a “soft” massless gluon. **Infrared Divergence**

Improved Parton Model

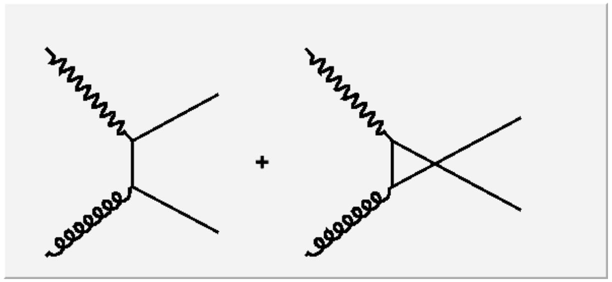
- $d\sigma/dp^2_T$ is also singular as $p^2_T \to 0$
- In the region $-\hat{t} \ll \hat{s}$, $d\sigma/dp^2_T$ represents the full p^2_T distribution of the final-state parton jets.



- In fact: in the diagram of the partonic model the p_T of the outgoing relative to the virtual photon always vanishes,



- while all the other diagrams are negligible in comparison with $\gamma^* q \to qg$ in the limit: $-\hat{t} \ll \hat{s}$
.



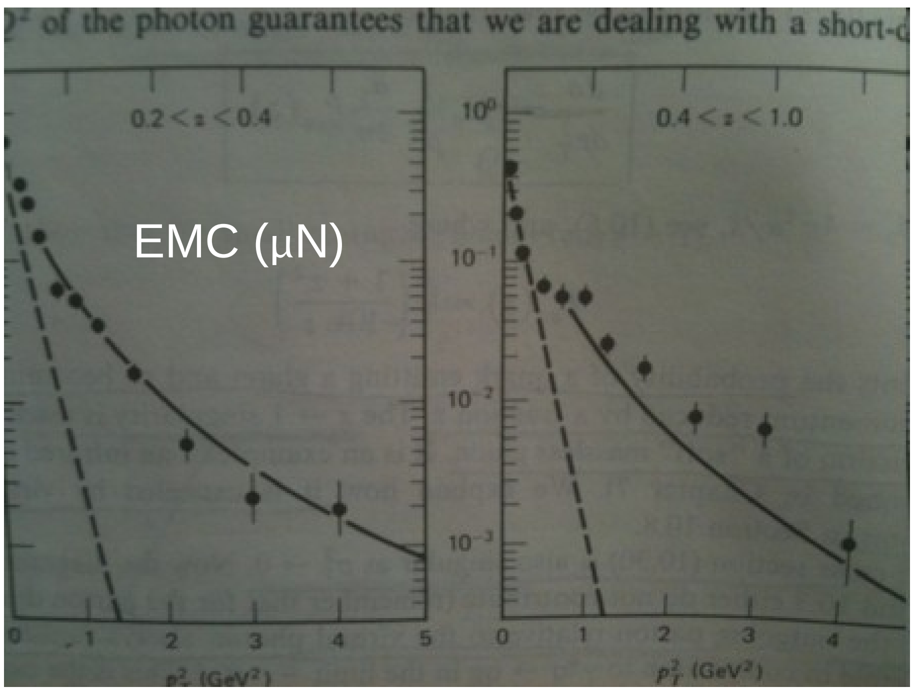
Improved Parton Model

- **Experimental signature?**

The presence of gluon emission is signalled by a quark-jet and a gluon-jet in the final state, neither of which is moving along the direction of the γ^*:

$$p_T(\text{quark-jet}) \neq 0; \; p_T(\text{gluon-jet}) \neq 0$$

$d\sigma/\sigma dp^2_T\,(\mathrm{GeV}^{-2})$



$p^2_T\,(\mathrm{GeV}^2)$

In a parton model without g: all final-state jets would be collinear with the γ^*, with a spread of p_T of ~ 300 MeV (as required from the Uncertainty Principle for confined quarks)

Further proofs in:
e+e- → hadrons;
$pp \to$ large- p_T hadrons

Reminder: the large Q^2 of the $\gamma^* \to \alpha_s$ is small

Scaling violation. Dokhishitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) Equation

- How to include the gluon-strahlung diagram in the structure functions:

$$\hat{\sigma}(\gamma^* q \to qg) = \int_{\mu^2}^{\hat{s}/4} dp_T^2 \frac{d\hat{\sigma}}{d\,p_T^2} \simeq e_i^2 \hat{\sigma}_0 \int_{\mu^2}^{\hat{s}/4} \frac{dp_T^2}{p_T^2} \frac{\alpha_s}{2\pi} P_{qq}(z)$$

$$\simeq e_i^2 \hat{\sigma}_0 \left(\frac{\alpha_s}{2\pi} P_{qq}(z) \ln\left(\frac{\hat{s}/4}{\mu^2}\right) \right) \simeq e_i^2 \hat{\sigma}_0 \left(\frac{\alpha_s}{2\pi} P_{qq}(z) \ln\left(\frac{Q^2}{\mu^2}\right) \right)$$

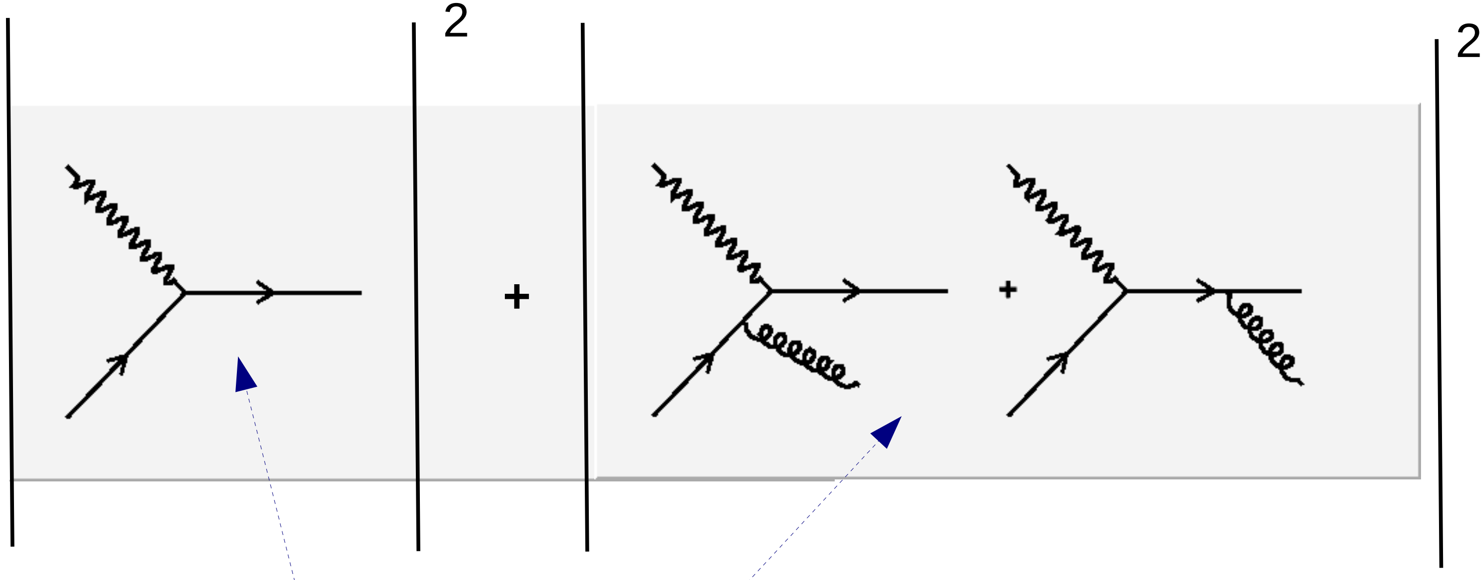
- where $\left(p_T^2\right)_{max} = \frac{\hat{\sigma}}{4} = Q^2 \frac{1-z}{4z}$; and $\ln\left(\frac{\hat{s}}{4}\right) \simeq \ln\left(Q^2\right)$ in the large Q^2 limit
- $\mu \equiv$ cutoff to regularize the divergence when $p^2{}_T \to 0$

$p^2{}_T$ limit of the gluon

Scaling violation. DGLAP Equation

- Adding everything one has:

$$\frac{F_2(x)}{x} =$$



$$= \sum_q e_q^2 \int_x^1 \frac{dy}{y} q(y) \left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{y}\right) \ln\left(\frac{Q^2}{\mu^2}\right) \right]$$

y for the incoming parton
x for the outgoing parton
y > x

- term which depends on Q^2
- **Scaling violation !**
- In QCD: F_2 depends on x and Q^2: the scaling violation is only logarithmic
- The violation of the Bjorken scaling is a signature of the gluon emission

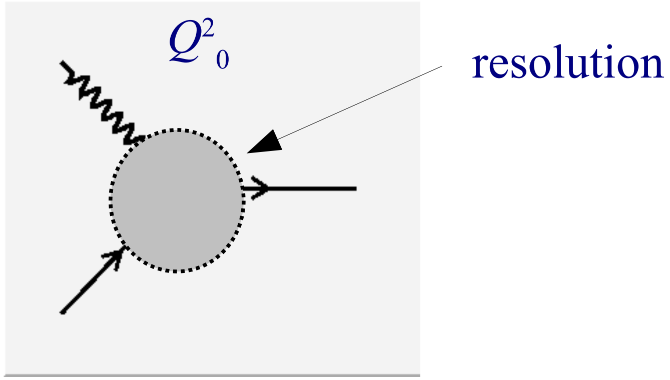
Scaling violation. DGLAP Equation

- The formula can be written also in this manner:

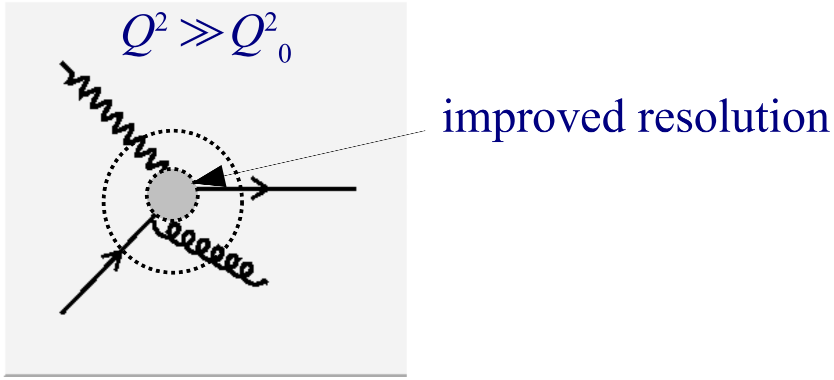
$$\frac{F_2(x,Q^2)}{x} = \sum_q e_q^2 \int_x^1 \frac{dy}{y}\left(q(y)+\Delta q(y,Q^2)\right)\delta\left(1-\frac{x}{y}\right)$$
$$= \sum_q e_q^2 \left(q(x)+\Delta q(x,Q^2)\right)$$

with $\Delta q(x,Q^2)=\frac{\alpha_s}{2\pi}\ln\left(\frac{Q^2}{\mu^2}\right)\int_x^1 \frac{dy}{y} q(y) P_{qq}\left(\frac{x}{y}\right)$

- **The quark density depends on Q^2**: A γ^* with larger Q^2 has a higher resolution and see more partons within the proton.
  More probability of finding a quark at small x and less probability of finding a quark at high x due the gluon emission.



$\gamma^*(Q^2_0)$ sees $q(x)$



$\gamma^*(Q^2)$ sees softer quarks inside $q(x)$

Scaling violation. DGLAP Equation

- The Q^2 evolution of the $q(x,Q^2)$ is determined by QCD through:

$$\Delta q(x,Q^2) = \frac{\alpha_s}{2\pi} \ln\left(\frac{Q^2}{\mu^2}\right) \int_x^1 \frac{dy}{y} q(y) P_{qq}\left(\frac{x}{y}\right)$$

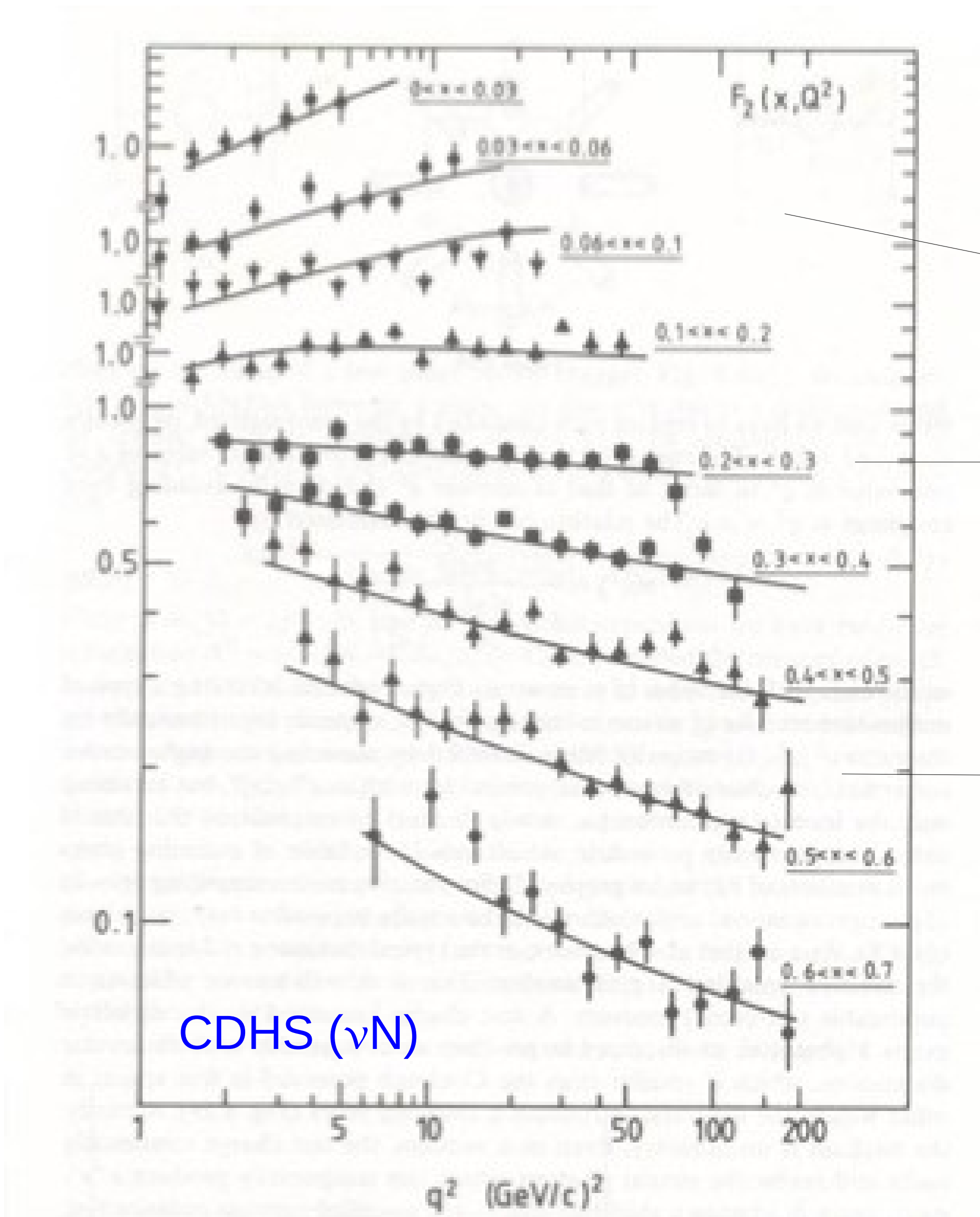
which can be written:

$$\boxed{\frac{d\,q(x,Q^2)}{d\ln(Q^2)} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} q(y,Q^2) P_{qq}\left(\frac{x}{y}\right)}$$

DGLAP evolution equation

- A q with momentum fraction x (on the left-hand side) could have come from a parent q with a larger momentum fraction $y > x$ (on the right-hand side) which has radiated a gluon.
The probability that this happens is proportional to $\alpha_s\, P_{qq}(x/y)$.
The integral is the sum over all possible momentum fractions y ($> x$) of the parent q.

Scaling violation. DGLAP Equation



- Given the experimental value $q(x,Q^2{}_0)$ it is possible to calculate its value for any Q^2 through the use of the DGLAP equation.

$x < 0.25$: F_2 increases with Q^2

$x = 0.25$: F_2 is independent of Q^2
Bjorken scaling

$x > 0.25$: F_2 decreases with Q^2

Increasing the Q^2 we resolve an increasing number of " soft" quarks.
The large-momentum quark component $(x \simeq 1)$ is depleted and shifted towards low momentum $(x \simeq 0)$.

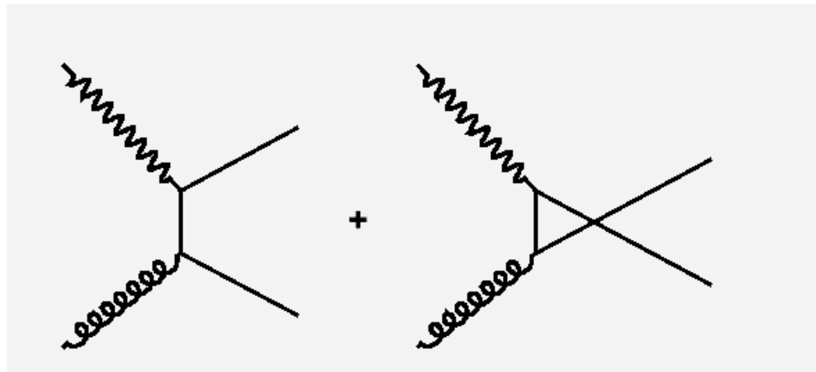
Scaling violation. DGLAP Equation

- We have assumed α_s constant. The result is valid also when $\alpha_s(p^2_T)$.
- We are working in a kinematic region with two large quantities: p^2_T , Q^2 and the dominant region is $p^2_T \ll Q^2$.

  In this limit, the study of the higher orders introduces p^2_T as the argument of α_s

Gluon Pair Production

- $\gamma^* g \rightarrow qq$

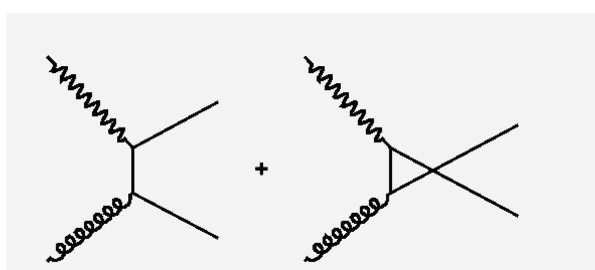


Compton diagram

$$\overline{|M^2|} = 32\pi^2 (e_q^2 \alpha \alpha_s) \frac{1}{2} \left(\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - \frac{2\hat{s}Q^2}{\hat{t}\hat{u}} \right)$$

color factor: $\frac{1}{2}\frac{3\times 3-1}{8} = \frac{1}{2}$

- New contribution to $F_2(x,Q^2)/x$:

$\frac{F_2(x)}{x} = \Bigg|$  $\Bigg|^2$

$$= \sum_q e_q^2 \int_x^1 \frac{dy}{y} g(y) \frac{\alpha_s}{2\pi} P_{qg}\left(\frac{x}{y}\right) \ln\left(\frac{Q^2}{\mu^2}\right)$$

$g(y) \equiv$ gluon density of the proton

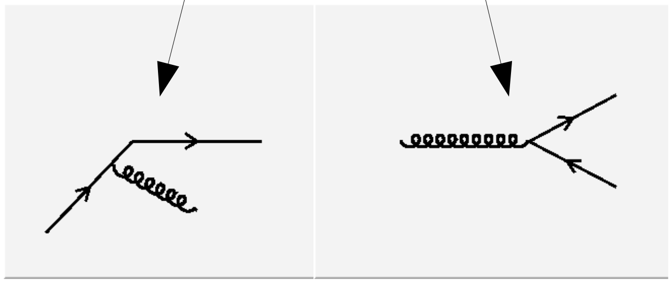
Gluon Pair Production

- $P_{qg}(z) = \frac{1}{2}\left(z^2+(1-z)^2\right)$ probability that a gluon annihilates $g \rightarrow q\,\bar{q}$ such that the q has a fraction z of its momentum

Complete Evolution Equations for the Parton Densities

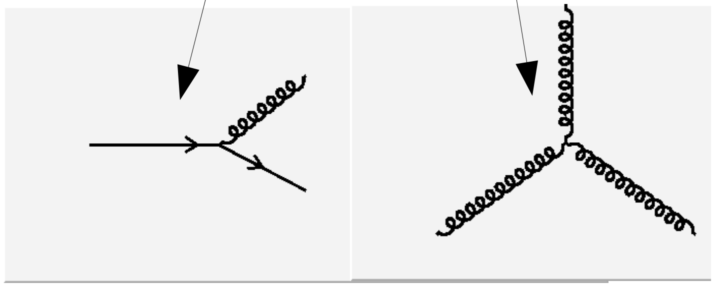
- $$\frac{d\,q_i(x,Q^2)}{d\log Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y}\left(q_i(y,Q^2)\,P_{qq}\left(\frac{x}{y}\right)+g(y,Q^2)\,P_{qg}\left(\frac{x}{y}\right)\right)$$

Quark evolution equation



$$\frac{d\,g(x,Q^2)}{d\log Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y}\left(\sum_i q_i(y,Q^2)\,P_{gq}\left(\frac{x}{y}\right)+g(y,Q^2)\,P_{gg}\left(\frac{x}{y}\right)\right)$$

Gluon evolution equation



$\sum \equiv$ the sum runs over quarks and antiquarks of all flavours
P_{gq} does not depend on the index i if the quark masses can be neglected

Complete Evolution Equations for the Parton Densities

- $$P_{gq}(z) = \frac{4}{3}\frac{1+(1-z)^2}{z} \qquad P_{gg}(z) = 6\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right)$$

Physical interpretation of the splitting functions

- ***The parton densities are universal, are indipendent from the specific process (DIS in this case) used for their measurement.***
- ***Their evolution as function of Q^2 is predicted by QCD through the use of the splitting function: P_{qq}, P_{gg}, P_{gq}, P_{qg}***
- One can write:

$$q(x,Q^2)+\Delta q(x,Q^2) = \int_0^1 dy \int_0^1 dz\, q(y,Q^2)\,\wp_{qq}(z,Q^2)\,\delta(x-zy)$$

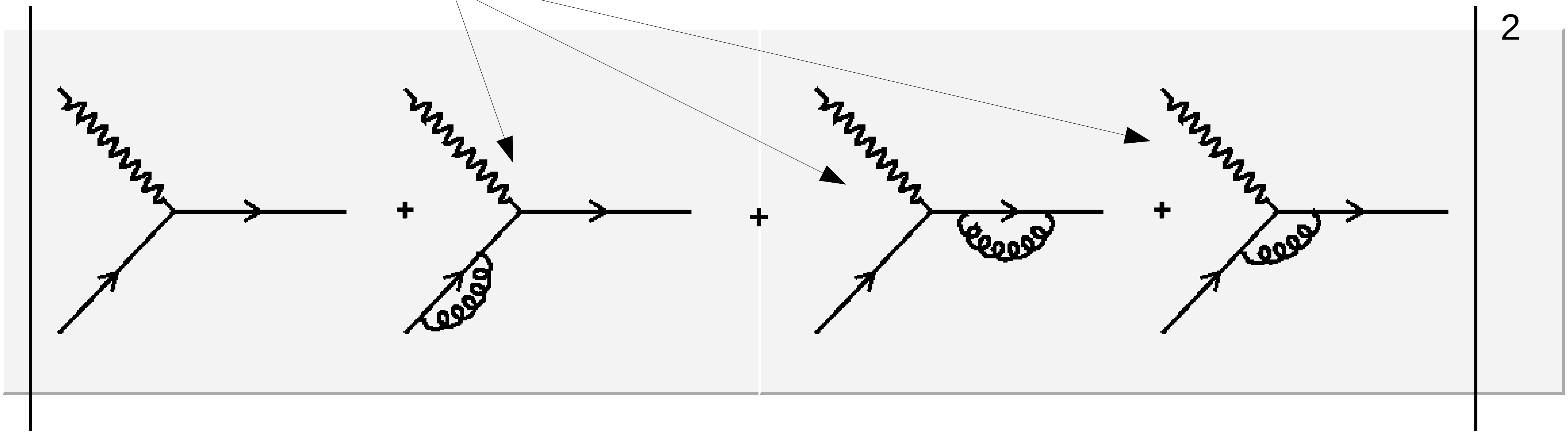
where:

$$\wp_{qq}(z,Q^2) \equiv \underbrace{\delta(1-z)}_{\text{no-change}}+\frac{\alpha_s}{2\pi}\underbrace{P_{qq}(z)\ln\left(\frac{Q^2}{\mu^2}\right)}_{\text{gluon emission}}$$

probability density of finding a quark inside a quark with fraction *z* of the parent quark momentum to first order in α_s

Physical interpretation of the splitting functions

- There are also other graphs with the same initial and final state as the process $\gamma^* q \rightarrow q$:



The new graphs give contributions of $O(\alpha\alpha_s)$ coming from the interference with the partonic diagram.
These interference contributions are singular at $z = 1$.
Such singolarity cancel exactly the singularity at $z = 1$ present in:

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$

Physical interpretation of the splitting functions

- Giving a look in a different manner, it has to be:

$$1 = \int_0^1 \wp_{qq}(z, Q^2)$$

that is the probability of finding a quark in a quark integrated over all z must add up to 1.
But from:

$$\wp_{qq}(z, Q^2) \equiv \delta(1-z) + \frac{\alpha_s}{2\pi} P_{qq}(z) \ln\left(\frac{Q^2}{\mu^2}\right)$$

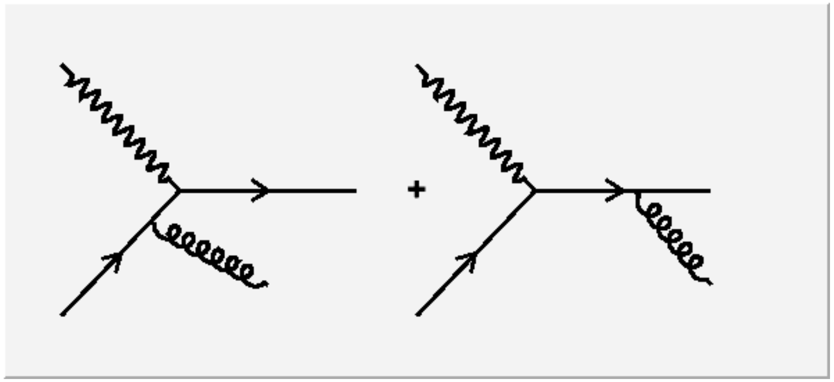
to have an integral equal to 1, it must be that:

$$\int_0^1 P_{qq}(z)\, dz = 0$$

The virtual diagrams regularize the divergences present in the integrals.

Physical interpretation of the splitting functions

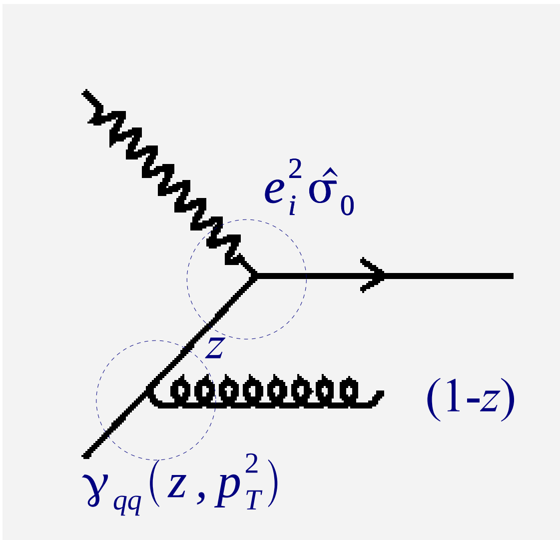
- Notes:



In our interpretation (using the splitting function *P*) we discussed only the first diagram. The gluon can be considered as part of the proton structure.
But:
1) both diagrams are required to ensure gauge invariance of the amplitude;
2) the second only plays the role of canceling the contributions from the unphysical polarization states of the gluon. Adopting a physical gauge, in which we sum only over transverse gluons, only the first diagram remains.

The Weizsäcker - Williams formula

- $\gamma^* g \to qg$

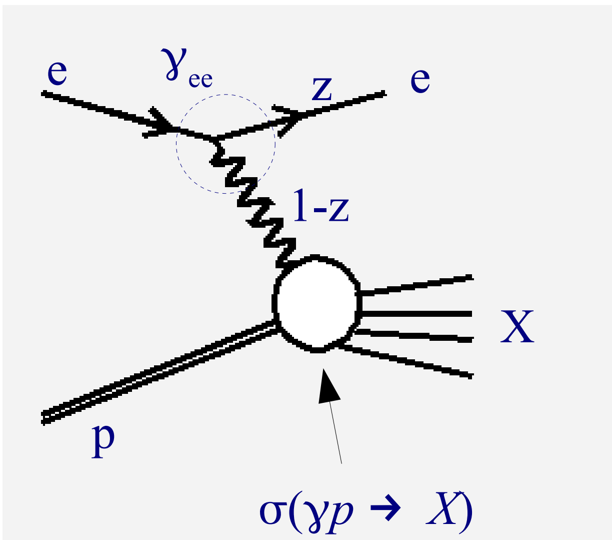


$$\frac{d\hat{\sigma}}{dz\,dp_T^2} = (e_i^2 \hat{\sigma}_0)\left(\frac{\alpha_s}{2\pi}\frac{1}{p_T^2}P_{qq}(z)\right) = (e_i^2\hat{\sigma}_0)\gamma_{qq}(z, p_T^2)$$

The $O(\alpha\alpha_s)$ $\sigma(\gamma^* g \to qg)$ factors into the $O(\alpha)$ parton model cross section ($e_i^2 \hat{\sigma}_0$) and the probability ($\gamma_{qq}(z, p_T^2)$) that the quark radiates a gluon with fraction $(1-z)$ of its momentum and with transverse momentum p_T

The Weizsäcker - Williams formula

- Such factorization is not special to quarks and gluons (QCD), but it was known since the work of Weiszäcker and Williams, 1934, that it applies equally well to leptons and photons (QED). Example: ***ep* → *eX***



$$\frac{d\sigma}{d\,p_T^2} = \gamma_{ee}(z, p_T^2)\left[\sigma(\gamma\, p \to X)\right]_{E_\gamma=(1-z)E}$$

$$\gamma_{ee}(z, p_T^2) = \frac{\alpha}{2\pi}\frac{1}{p_T^2}P_{ee}(z)$$

z, p_T are, respectively, the momentum fraction and transverse momentum of the outgoing electron.

$$P_{ee}(z) = \frac{1+z^2}{1-z}$$

$\gamma_{ee} \equiv$ equivalent γ distribution

$$P_{qg} \to P_{e\gamma} = z^2 + (1-z)^2$$

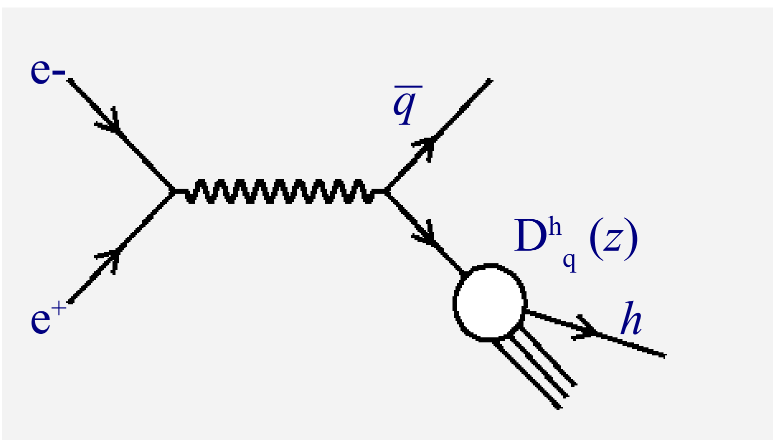
Fragmentation functions and their scaling properties

- We start , from the process $e^+e^- \rightarrow X$
  The formula:

$$\sigma(e^+e^- \to hadrons) = \sum_q \sigma(e^+e^- \to q\bar{q})$$

is valid under the hypothesis that the quarks have to fragment into hadrons with probability equal to 1.

- To describe the fragmentation of $q \rightarrow$ hadrons, it is possible to use a formalism analogous to that of the q inside the hadrons:



Observation of a hadron h whose energy is measured to be E_h

$$\frac{d\sigma}{dz}(e^+e^- \to h X) = \sum_q \sigma(e^+e^- \to q\bar{q})\left[D_q^h(z) + D_{\bar{q}}^h(z)\right]$$

Fragmentation functions and their scaling properties

- The process happens as two sequential events:
  1) production of a $q\,\bar{q}$ pair
  2) hadrons fragmentation

- The fragmentation function *D* represents the probability that the hadron *h* is found in the quark or antiquark remnants, carrying a fraction *z* of its energy:

$$z \equiv \frac{E_h}{E_q} = \frac{E_h}{E_{beam}} = \frac{2E_h}{Q}$$

- ***D*(*z*)** describes **parton → hadron**
  ***f*(x)** describes **hadron → parton**

- As for the *f*:

$$\sum_h \int_0^1 z\, D_q^h(z)\, dz = 1 \longleftrightarrow$$

Momentum (or energy) conservation
clearly the same relation holds for: $D_{\bar{q}}^h(z)$

$$\sum_q \int_{z_{min}}^1 \left[D_q^h(z) + D_{\bar{q}}^h(z) \right] dz = n_h$$

$z_{min} = 2m_h/Q$ threshold energy to produce a hadron of mass m_h

Fragmentation functions and their scaling properties

- $$\sum_q \int_{z_{min}}^{1} \left[D_q^h(z) + D_{\bar{q}}^h(z) \right] dz = n_h$$

The equation says that the number n_h of hadrons of type h is given by the sum of probabilities of obtaining h from all possible parents, namely, from quarks or antiquarks of any flavor.

- Usually the $D^h_q(z)$ are often parametrized in such manner:

$$D_q^h(z) = N \frac{(1-z)^n}{z} = (n+1) < z > \frac{(1-z)^n}{z}$$

n, N are costants

Moreover for two-jet events: $n_h \sim \ln\left(\frac{Q/2}{m_h}\right)$

Fragmentation functions and their scaling properties

- One has:

$$\frac{1}{\sigma}\frac{d\sigma}{dz}\left(e^{+}e^{-}\rightarrow hX\right) = \frac{\sum e_q^2\left[D_q^h(z)+D_{\bar{q}}^h(z)\right]}{\sum_q e_q^2} = \mathscr{F}(z)$$

σ and $d\sigma/dz$ depend on the annihilation energy Q, their ratio is independent on Q and dependent only on z.
Such scaling result is not a complete surprise, because we have relied on the scaling parton model.

Fragmentation functions and their scaling properties

Scaling of the fragmentation functions

But there are **scaling violation:** $\mathscr{F}(z) \rightarrow \mathscr{F}(z, Q^2)$ with dependence of Q^2 of the type $\ln Q^2$ in the low z region ($z < 0.2$)

The scaling violation at small z is not only due to gluon emission but also to the **production of heavy quarks** such as $\boldsymbol{c}$ and $\boldsymbol{b}$.

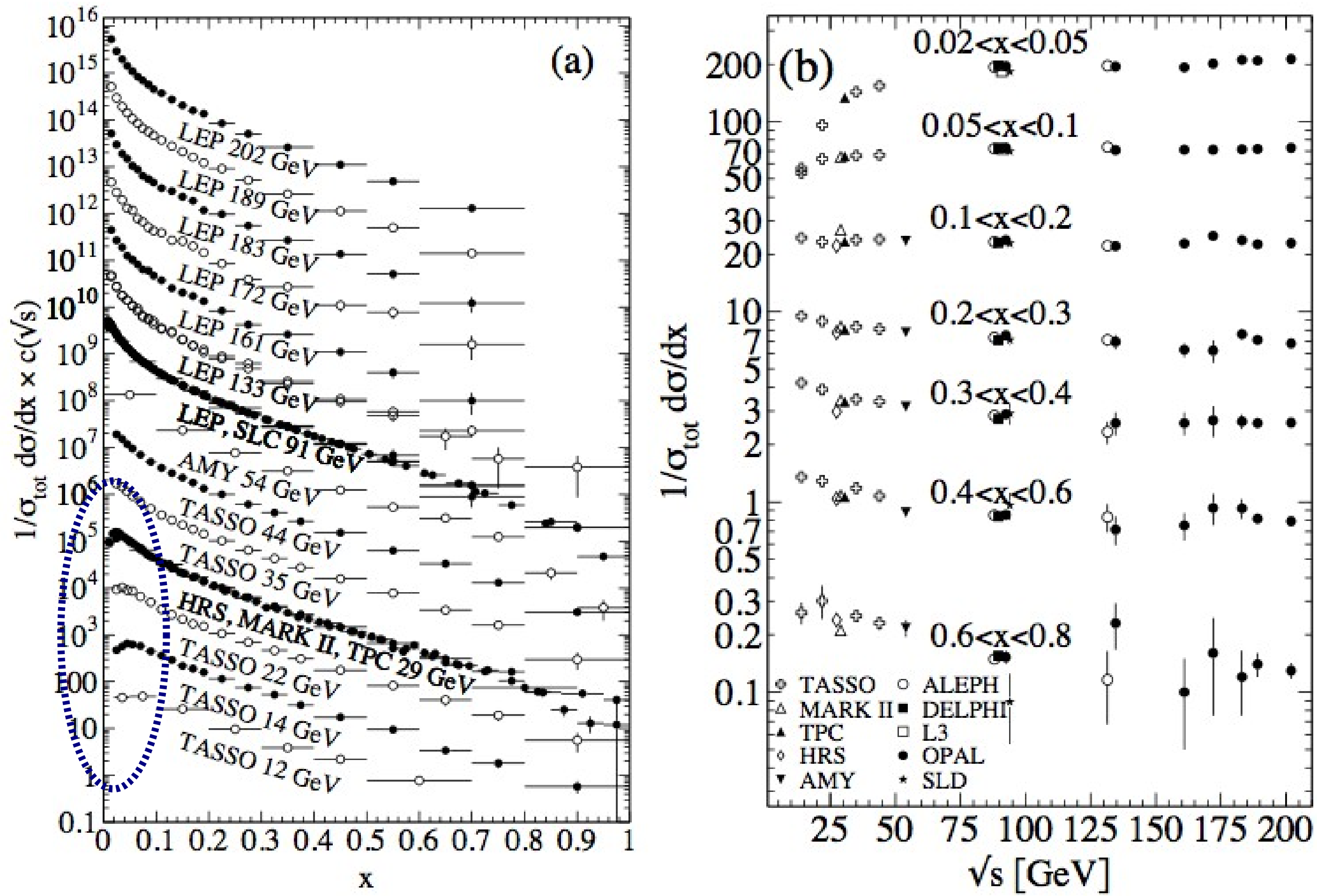


Figure 19.2: The e^+e^- fragmentation function for all charged particles is shown [8, 26–42] (a) for different CM energies $\sqrt{s}$ versus x and (b) for various ranges of x versus $\sqrt{s}$. For the purpose of plotting (a), the distributions were scaled by $c(\sqrt{s}) = 10^i$ with i ranging from $i = 0$ ($\sqrt{s} = 12$ GeV) to $i = 13$ ($\sqrt{s} = 202$ GeV).

Fragmentation functions and their scaling properties

- $D(z)$ describes general properties of the partons (independent of how the partons have been produced). For example:

$$\frac{1}{\sigma}\frac{d\sigma}{dz}(ep \rightarrow hX) = \frac{\sum e_q^2 f_q(x)\, D_q^h(z)}{\sum_q e_q^2 f_q(x)}$$