Quantum Chromodynamics (QCD)

Asymptotic freedom and Confinement

- Renormalization
- α_s and its energy dependence
- The running costant
- Relative size of finite order approximations
- Quark masses and thresholds
- Perturbative prediction of observables
- Renormalization scale dependence
- Nonperturbative QCD methods

Renormalization

- In quantum field theory like QCD and QED, physical quantities ***R*** can be expressed by a perturbation series in powers of α_s or α, respectively.
- If, for example, $\alpha_s \ll 1$, the series may converge sufficiently quickly such that it provides a realistic prediction of ***R*** even if only a limited number of perturbative orders will be known.
- Example of ***R*** : σ, Γ, jets production rates, hadronic event shapes, …

- Consider $\boldsymbol{R(Q^2, \alpha_s)}$ being **dimensionless** and depending on α_s and only a single energy scale ***Q***. $\boldsymbol{Q^2}$ larger than any other relevant, dimensional parameter such as quark masses. In the following: $m_q \equiv 0$

- When calculating ***R*** as a perturbative series in power of α_s, **ultraviolet divergences** occur.
- To give meaning to ***R*** the divergences are first made **temporarely finite** through the **regularization** procedure.
- The regularization introduces additional parameters: a finite mass to the gluons, a cutoff parameter Λ, etc.
- The divergences are then expressed in a well defined manner (even if with divergent limit).

Renormalization

- These divergencies now regularized are then removed absorbing them in the definition of physical parameters through a procedure called **renormalization**.
- The renormalization is a precise (even if arbitrary) prescription which introduces a **new energy scale μ**
- ***R*** and **α_s become function of this renormalization scale μ.**
- Being ***R*** dimensionless:

$$\boldsymbol{R} \equiv \boldsymbol{R}(Q^2/\mu^2, \alpha_s); \qquad \alpha_s \equiv \alpha_s(\mu^2)$$

- The choice of **μ is arbitrary**, but the value of the experimental observable ***R*** can't depend on **μ:**

$$\mu^2 \frac{d}{d\mu^2} \boldsymbol{R}(Q^2/\mu^2, \alpha_s) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right) \boldsymbol{R} = 0$$

- The equation implies that any explicit dependence of ***R*** on **μ** must be cancelled by an appropriate **μ**-dependence of α_s to all orders.

Renormalization

- Example:
  doing the natural identification: $Q^2 = \mu^2$
  → one eliminates the uncomfortable presence of a second and unspecified energy scale
  → α_s becomes a **running coupling constant**: $\alpha_s(Q^2)$
  → $\boldsymbol{R} = \boldsymbol{R}(\alpha_s(Q^2))$: the energy dependence of $\boldsymbol{R}$ enters only through the energy dependence of $\alpha_s(Q^2)$

- Any residual μ-dependence is a measure of the quality of a given calculation in finite perturbative order.

α_s and its energy dependence

- **While QCD does not predict the absolute size of α_s, its energy dependence is precisely determined.**
- If $\alpha_s(\mu^2)$ is measured at a given scale, QCD definitely predicts its size at any other energy scale Q^2 through the **renormalization group equation (RGE)**:

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \boxed{\beta(\alpha_s(Q^2))}$$

→ **beta function**

- The perturbative expansion of the β function is calculated to complete 4-loop approximation:

$$\beta(\alpha_s(Q^2)) = -\beta_0 \alpha_s^2(Q^2) - \beta_1 \alpha_s^3(Q^2) - \beta_2 \alpha_s^4(Q^2) - \beta_3 \alpha_s^5(Q^2) + O(\alpha_s^6)$$

where (N_f = n. of active quarks flavours at the energy scale Q, $C_A = 3$, $C_F = 4/3$ in $SU(3)_C$):

$$\beta_0 = \frac{11 C_A - 2 N_f}{12\pi} = \frac{33 - 2N_f}{12\pi} \quad \text{1-loop}$$

$$\beta_1 = \frac{17 C_A^2 - 5 C_A N_f 3 C_F N_f}{24\pi^2} = \frac{153 - 19 N_f}{24\pi^2} \quad \text{2-loop}$$

$$\beta_2 = \frac{77139 - 15099 N_f + 325 N_f^2}{3456\pi^3} \quad \text{3-loop}$$

$$\beta_3 = \frac{29243 - 6946 N_f + 405{,}089 N_f^2 + 1{,}49931 N_f^3}{256\pi^3} \quad \text{4-loop}$$

For theories exhibiting SU(N) symmetry the group constants are: :
$C_A = N$; $C_F = (N^2-1)/2N$

α_s and its energy dependence

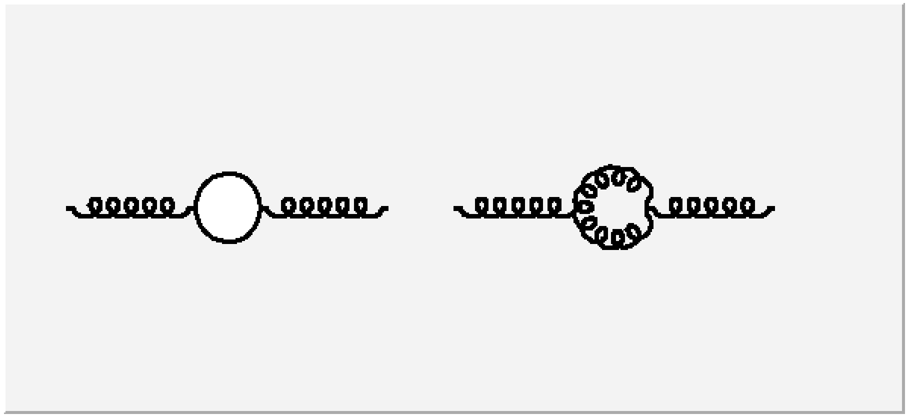
- β_0, β_1 are independent of the renormalization scheme.
- All higher order β coefficients are scheme dependent.

The running costant

- The 1-loop solution of:

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s(Q^2))$$

comes from:



$$Q^2 \frac{d\,\alpha_s(Q^2)}{d\,Q^2} = -\beta_0 \alpha_s^2(Q^2)$$

- and is:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) \ln\left(\frac{Q^2}{\mu^2}\right)}$$

- relationship between $\alpha_s(Q^2)$ and $\alpha_s(\mu^2)$
- if $\beta_0 > 0$ ($N_f < 17$) and $Q^2 \rightarrow \infty$; $\rightarrow$ $\alpha_s(Q^2) \rightarrow 0$ **asymptotic freedom**
- if $Q^2 \rightarrow 0$; $\rightarrow$ $\alpha_s(Q^2) \rightarrow \infty$ $\rightarrow$ **confinament**

The running costant

- Example:

  $\alpha_s(\mu^2 = M^2_Z) = 0.12$ per $N_f = 2, \ldots, 5$
  $\alpha_s(Q^2) > 1$ for $Q^2 \leq O(100 \text{ MeV} \ldots 1 \text{ GeV})$

- $Q^2 < 1 \text{ GeV}^2$ this is the non-perturbative region, where the confinament happens. Here the RGE equation cannot be used.

- Including the β_1 term or higher similar but more complicated expression are obtained. They can be solved numerically.

The running costant

- Putting:

$$\Lambda^2 = \frac{\mu^2}{e^{1/(\beta_0 \alpha_s(\mu^2))}}$$

one obtains:

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}$$

for $Q^2 \rightarrow \Lambda^2$; $\rightarrow$ $\alpha_s(Q^2) \rightarrow \infty$

Example: $\Lambda \sim 0.1$ GeV for $\alpha_s(M_Z = 90.2$ GeV $) = 0.12$ and $N_f = 5$

- This is the standard parameterization.

The running costant

- This parameterization has a certain number of caveats:
  1) $\alpha_s(Q^2)$ must be continuous when crossing a quark threshold →
     Λ depends on the quarks number
  2) Λ depends on the renormalization scheme

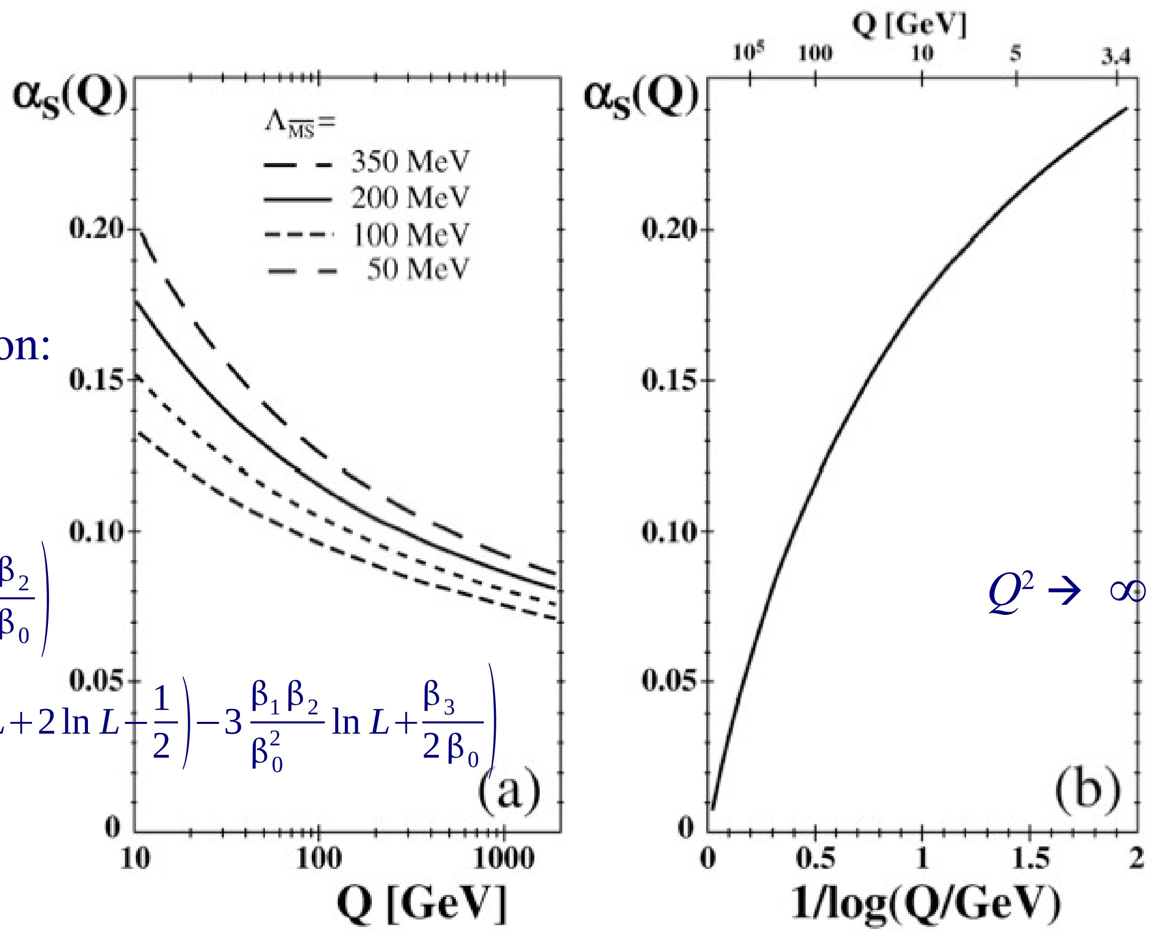
- Renormalization scheme used here : "***modified minimal subtraction scheme***": $\overline{MS}$
  Therefore Λ will be labelled: $\Lambda_{\overline{MS}}^{(N_f)}$

The running costant

In 4-loop approximation:

$$\alpha_s(Q^2) = \frac{1}{\beta_0 L} - \frac{1}{\beta_0^2 L^2}\beta_1 \ln L$$
$$+ \frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2 L - \ln L - 1\right)\frac{\beta_2}{\beta_0}\right)$$
$$+ \frac{1}{\beta_0^4 L^4}\left(\frac{\beta_1^3}{\beta_0^3}\left(-\ln^3 L + \frac{5}{2}\ln^2 L + 2\ln L - \frac{1}{2}\right) - 3\frac{\beta_1\beta_2}{\beta_0^2}\ln L + \frac{\beta_3}{2\beta_0}\right)$$

$$L = \ln(Q^2/\Lambda_{\bar{MS}}^2)$$



$Q^2 \rightarrow \infty \quad \alpha_s(Q^2) \rightarrow 0$

Figure 4: (a) The running of $\alpha_s(Q)$, according to equation 7, in 4-loop approximation, for different values of $\Lambda_{\overline{MS}}$; (b) same as full line in (a), but as function of 1/log(Q/GeV) to demonstrate asymptotic freedom, i.e. $\alpha_s(Q^2) \rightarrow 0$ for $Q \rightarrow \infty$.

The running costant

- Any experimental proof of asymptotic freedom will therefore require precise determination of α_s (or of other observables which depend on $\alpha_s(Q^2)$), in a possibly large range of energy scale.
- This range should include as small as possible energies, since the relative energy dependence is largest there.
- To date, precise experimental data and respective QCD analyses are available in the range of $Q \approx$ 1 GeV to 1000 GeV.

Relative size of finite order approximations

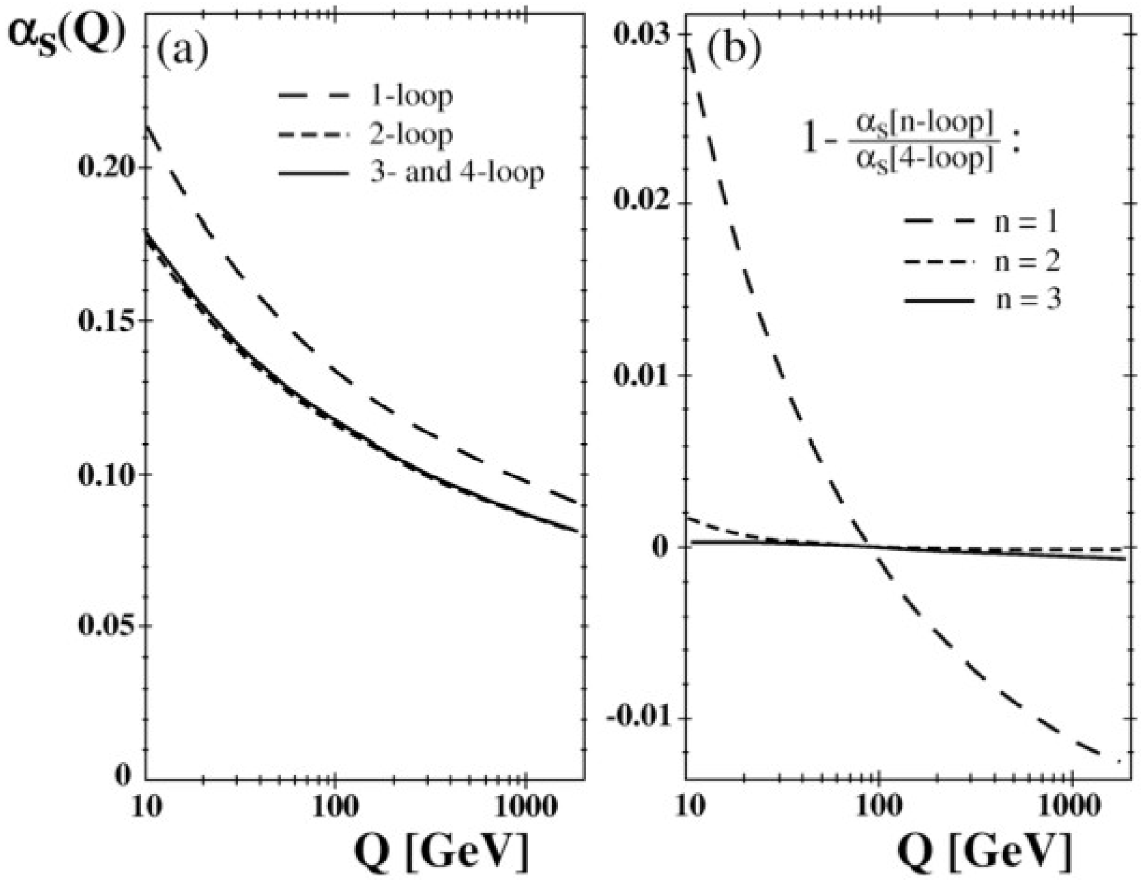


Figure 5: (a) The running of $\alpha_s(Q)$, according to equation 7, in 1-, 2- and 3-loop approximation, for $N_f = 5$ and the same value of $\Lambda_{\overline{MS}} = 0.22$ GeV. The 4-loop prediction is indistinguishable from the 3-loop curve. (b) Fractional difference between the 4-loop and the 1-, 2- and 3-loop presentations of $\alpha_s(Q)$, for $N_f = 5$ and $\Lambda_{\overline{MS}}$ chosen such that, in each order, $\alpha_s(M_{Z^0}) = 0.119$.

Quark masses and thresholds

- Up to now $m_q \approx 0$ because Q^2 e μ^2 are much larger than any other energy scale
- This is not enterely correct: QCD studies for $Q \sim 1.5$ GeV ($\sim m_c$) or $Q \sim 4.5$ GeV ($\sim m_b$)
- **$m_q \neq 0$ effects**
  1) **Finite quark masses alter the pertubative predictions of the *R* observables**
     a) explicit quark mass corrections in higher than leading perturbative order are available only for very few observables.
     b) phase space effects can often be studied using hadronization models and MC simulation techniques.
  2) **Any quark-mass dependence of *R* introduces a term**

$$\mu^2 \frac{\partial m}{\partial \mu^2} \frac{\partial R}{\partial m}$$

to the equation of $\boldsymbol{R} = \boldsymbol{R}(Q^2/\mu^2, \alpha_s(\mu^2), m)$:

$$\mu^2 \frac{dR}{d\mu^2} \rightarrow \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} + \mu^2 \frac{\partial m}{\partial \mu^2} \frac{\partial}{\partial m} \right) R = 0$$

Quark masses and thresholds

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} - \gamma_m(\alpha_s)m\frac{\partial}{\partial m}\right)R(Q^2/\mu^2,\alpha_s,m/Q)=0$$

can be solved introducing a **running mass**, $m(Q^2)$, in a similar way as the running coupling $\alpha_s(Q^2)$ was obtained:

$$Q^2\frac{\partial m}{\partial Q^2} = -\gamma_m(\alpha_s)m(Q^2)$$

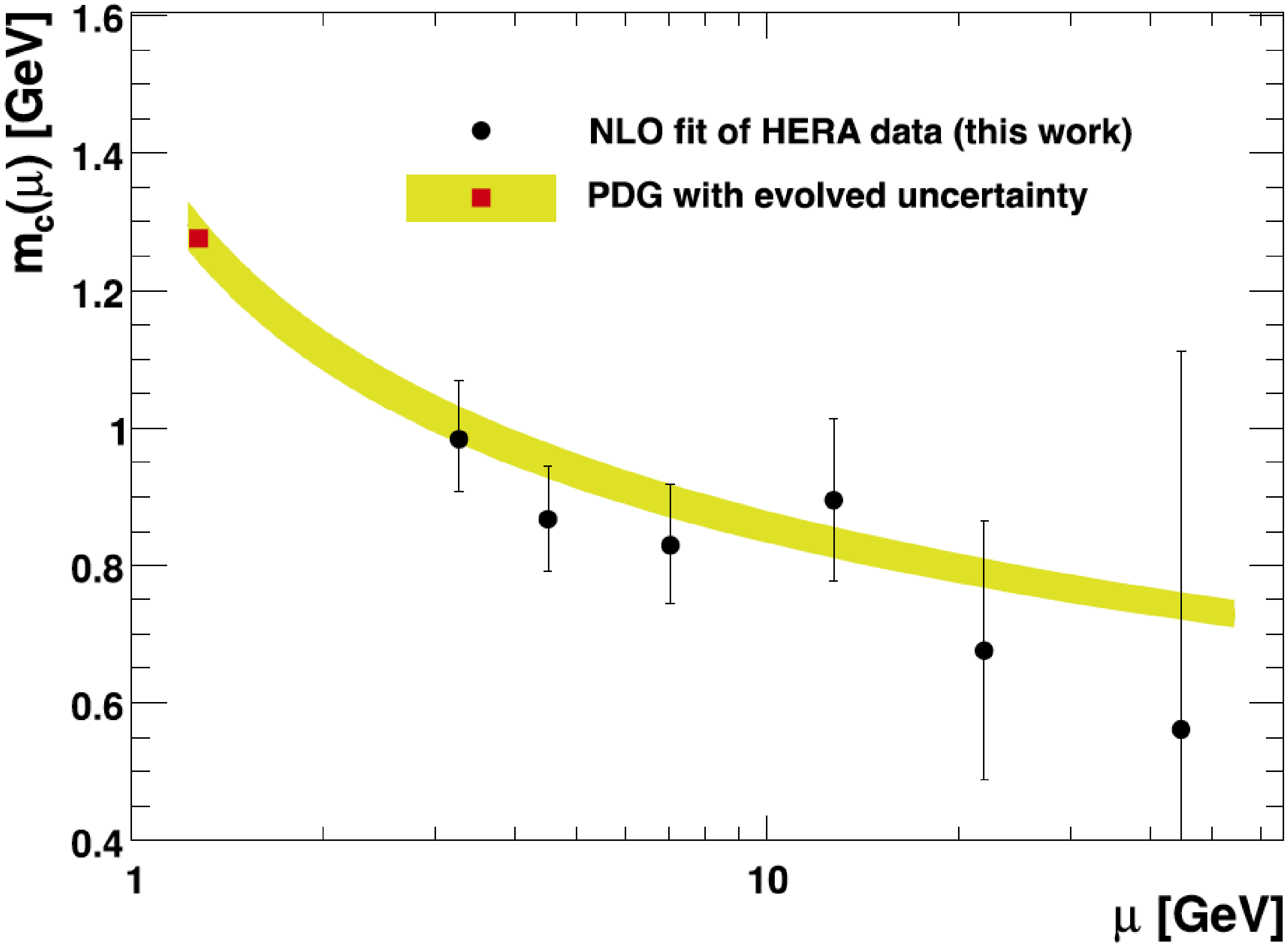
$\gamma_m(\alpha_s)$: **mass anomalous dimension**.

The solution is:

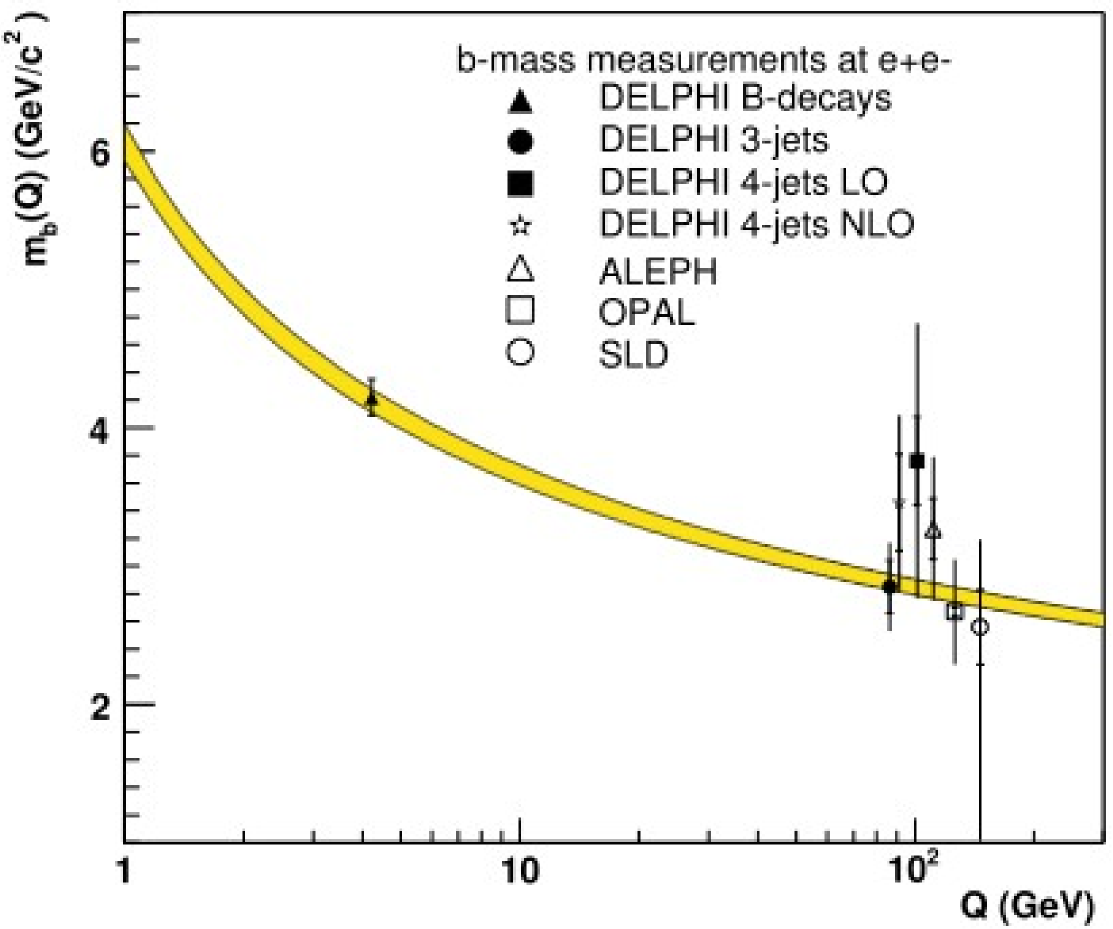
$$m(Q^2)=m(\mu^2)\exp\left[-\int_{\mu^2}^{Q^2}\frac{dQ^2}{Q^2}\gamma_m(\alpha_s(Q^2))\right]$$

$$=m(\mu^2)\exp\left[-\int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)}d\alpha_s\frac{\gamma_m(\alpha_s)}{\beta(\alpha_s)}\right]$$

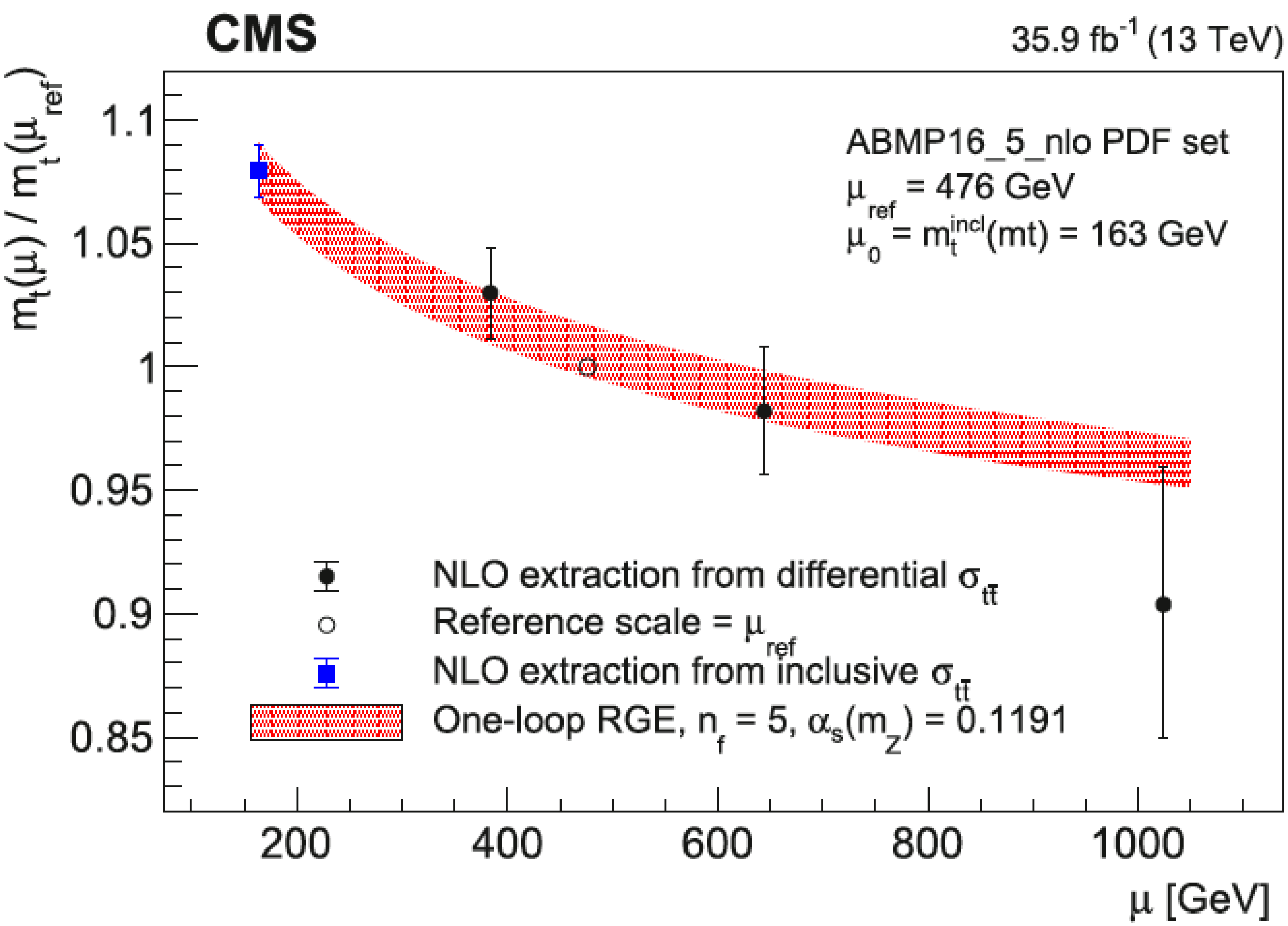
Quark masses and thresholds



Quark masses and thresholds



Quark masses and thresholds



Quark masses and thresholds

3) α_s indirectly depends on m_q through the dependence of the β coefficients on N_f

An effective theory for (N_f-1) flavours has to be consistent with a theory for N_f flavours at the heavy quark threshold: $\mu^{(Nf)} \sim O(m_q)$ → **matching conditions for the α_s values of the (N_f-1)- and N_f-quark flavour theories.**

The matching conditions are simple: $\alpha_s^{(Nf-1)} = \alpha_s^{(Nf)}$ for leading and next-to-leading orders

For higher orders the matching conditions are more complicated.

Perturbative predictions of observables

- α_s not directly observable, but only through ***R*** observables

- the observables are usually given by a power series of $\alpha_s(\mu^2)$:

$$\boldsymbol{R}(Q^2) = P_l \sum_n R_n \alpha_s^n$$
$$= P_l\left(R_0 + R_1 \alpha_s(\mu^2) + R_2(Q^2/\mu^2)\, \alpha_s^2(\mu^2) + \ldots\right)$$

R_n = n_{th} order coefficients of the perturbative series

$P_l R_0$ = the lowest-order value of ***R***

- For processes which involve gluons already in lowest order perturbation theory, P_l itself may include powers of α_s

  For example: $\Gamma(\Upsilon \rightarrow ggg \rightarrow \text{hadrons})$, $P_l \propto \alpha_s^3$

- If no gluons are involved in lowest order, for example: DIS processes, e+e- → *hadrons*,

  $P_l R_0$ = CONSTANT, and the usual choice of normalization is $P_l \equiv 1$

  R_0 = **lowest order** coefficient

  R_1 = **leading order** (LO) coefficient

  R_2 = **next-to-leading** (NLO) coefficient

  R_3 = **next-to-next-to-leading** (NNLO) coefficient

Perturbative predictions of observables

- QCD calculations in NLO perturbative calculation are available for many observables: event shapes, jet production rates, scaling violations of structure functions

- Calculations including the complete NNLO are available for some totally inclusive quantities: moments and sum rules of structure functions in DIS, $\Gamma(e^+e^- \rightarrow \text{hadrons})$, $\Gamma(Z^0 \rightarrow \text{hadrons})$, $\Gamma(\tau \rightarrow \text{hadrons})$, … but also for the production of vector bosons, Higgs bosons, Higgs in association with a vector boson, top-antitop couples, etc.

- This situation is due to the complicated nature of QCD:
gluon self-coupling → large number of Feynman diagrams in higher orders of perturbative theory.

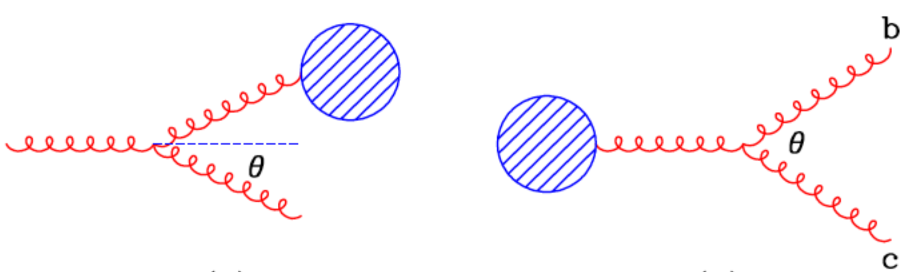
Perturbative predictions of observables

- Another approach to calculating higher order corrections is based on the **resummation of logarithms** which arise from soft and collinear singularities in gluon emission (*):

$$\boldsymbol{R} \approx A\,\alpha_s^n \log^{n+1}(\ldots) + B\alpha_s^n \log^n(\ldots) + C\,\alpha_s^n \log^m(\ldots)$$

$\alpha_s^n \log^{n+1}(\ldots) \equiv$ leading logarithms (LL)
$\alpha_s^n \log^n(\ldots) \equiv$ next-to-leading log (NLL)
$\alpha_s^n \log^m(\ldots) \equiv$ subdominant logarihtmic correction $0 < m < n$

(*) Also at high energy, short distances, the long distance aspects of QCD are not negligible. Soft and collinear gluon emission produces **infrared singularities** in perturbative theory. The light quarks ($m_q \ll \Lambda$) produce divergences when $m_q \to 0$ (**mass singularities**).



(a) Spaceline branching

(b) Timeline branching

Renormalization scale dependence

- ***R*** independence from the choice of μ^2:

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right) R = 0$$

- Putting $\alpha_s \rightarrow \alpha_s(\mu^2)$, and $Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s(Q^2))$, the pertubative expansion of ***R*** and of the β-function gives:

$$0 = \mu^2 \frac{\partial R_0}{\partial \mu^2} + \alpha_s(\mu^2) \mu^2 \frac{\partial R_1}{\partial \mu^2} + \alpha_s^2(\mu^2)\left[\mu^2 \frac{\partial R_2}{\partial \mu^2} - R_1 \beta_0\right]$$
$$+ \alpha_s^3(\mu^2)\left[\mu^2 \frac{\partial R_3}{\partial \mu^2} - \left[R_1 \beta_1 + 2 R_2 \beta_0\right]\right]$$
$$+ O(\alpha_s^4)$$

- To solve this equation, the $\alpha_s^n(\mu^2)$ coefficients have to go to zero for every *n*:

$$R_0 = costant;$$
$$R_1 = costant;$$
$$R_2(Q^2/\mu^2) = R_2(1) - \beta_0 R_1 \ln(Q^2/\mu^2);$$
$$R_3(Q^2/\mu^2) = R_3(1) - \left[2R_2(1) + R_1 \beta_1\right] \ln(Q^2/\mu^2) + R_1 \beta_0^2 \ln^2(Q^2/\mu^2)$$

Renormalization scale dependence

- Invariance of the complete perturbation series from μ^2 implies that R_2, R_3, … explicitly depend on μ^2
- In infinite order, the renormalization scale dependence of α and R_n cancel.
- In any finite (truncated) order the cancellation is not perfect.
- All realistic pQCD predictions include an explicit dependence on μ^2
- This dependence is more pronounced in LO, because R_1 does not depend on μ → no cancellation of the logarithm scale dependence of $\alpha_s(\mu^2)$.
- At NLO or at higher orders the dependence is weaker → partial cancellation due to the dependence of R_n (for $n \geq 2$) on μ^2
- The dependence on μ^2 is often used to test the theoretical calculations and the predictions of the observables.
  For example: e^+e^- → hadrons, the central value of $\alpha_s(\mu^2)$ is calculated for $\mu^2 = E^2_{cm}$, the changes of the result, obtained varying μ^2 in a reasonable interval, are taken as systematic uncertainties.

Nonperturbative QCD methods

- At large distances or low momentum transfers → $\alpha_s > 1$ → pQCD is not valid
- Non perturbative methods for $Q^2 < 1$ GeV^2 important to understand:
  fragmentation of q and g in hadrons (*hadronization*),
  absolute masses,
  splitting mass of the mesons
  ...
- **Hadronization models**: used in MC simulations to describe the transition of q and g into hadrons. They are based on QCD-inspired mechanisms: *string fragmentation*, *cluster fragmentation*.
  Many free parameters, adjusted in order to reproduce experimental data.
  Important tools for detailed QCD studies and to define resolution and acceptance of the detectors.

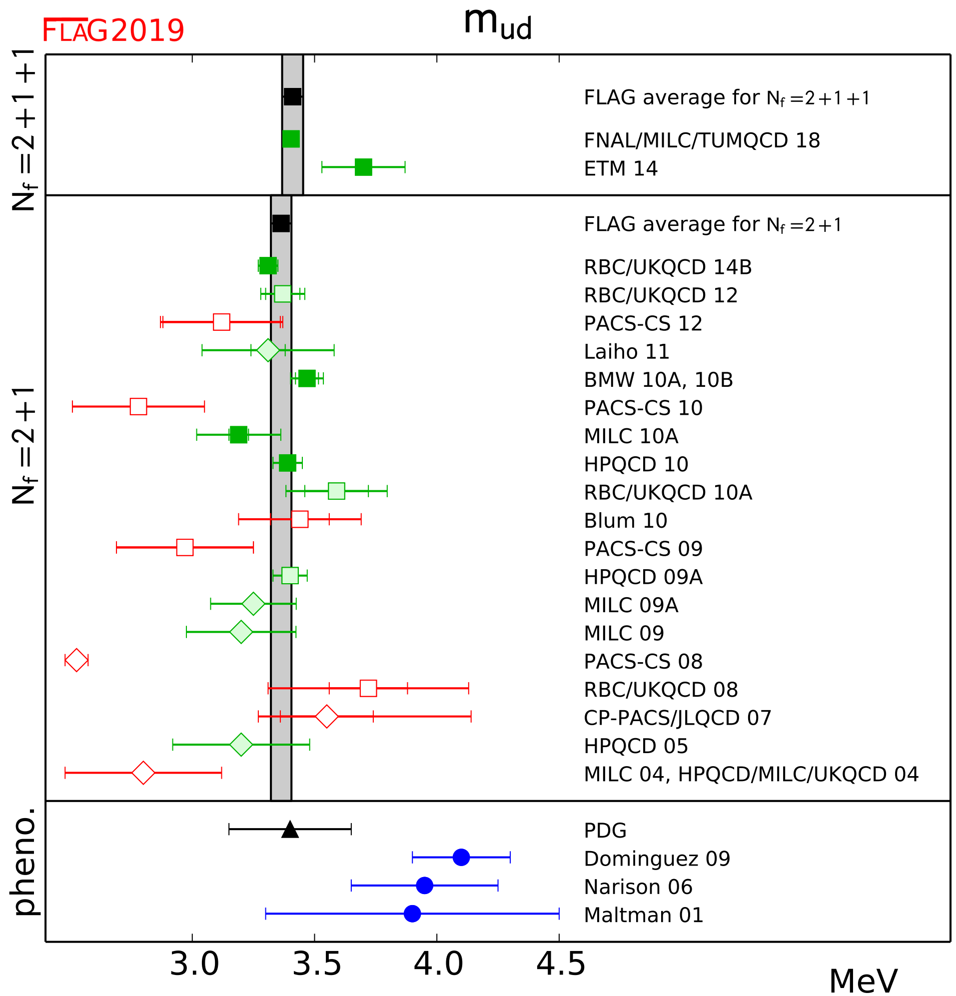
- **Power corrections**: analytic approach to approximate nonperturbative hadronization effects by means of perturbative methods, introducing a universal, non-perturbative parameter:

$$\alpha_0(\mu_I) = \frac{1}{\mu_I} \int_0^{\mu_I} dk\, \alpha_s(k)$$

  in such a way one parameterizes the unknown behaviour of $\alpha_s(Q^2)$ below a certain infrared matching scale μ_I

Nonperturbative QCD methods

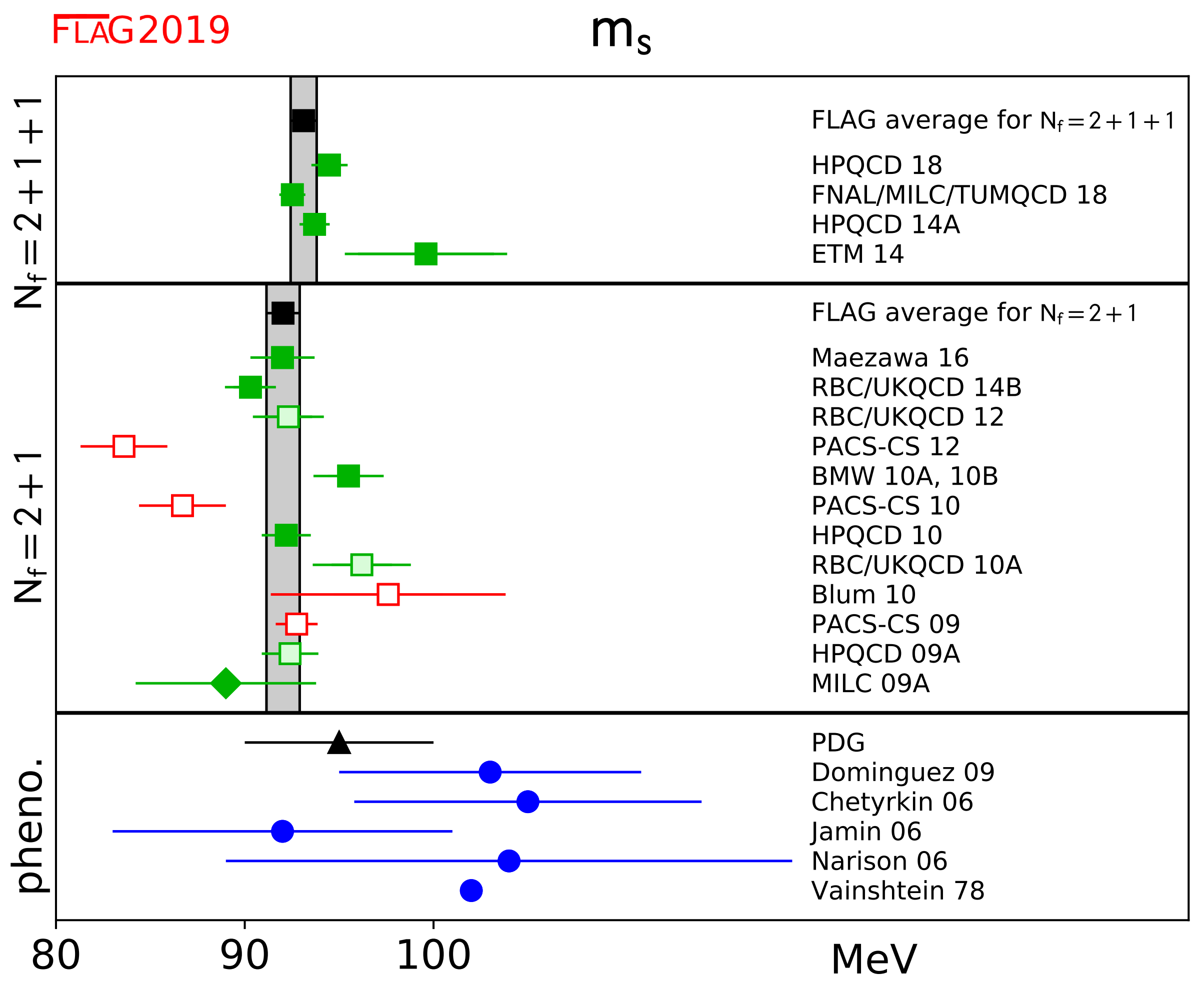
- **Lattice Gauge Theory**: field operators are applied on a discrete, 4-dimensional Euclidean space-time of hypercubes of side length ***a***
  It is used to calculate: hadron masses, mass splittings, QCD matrix elements.



$$m_{ud} = (m_u + m_d)/2$$

Nonperturbative QCD methods

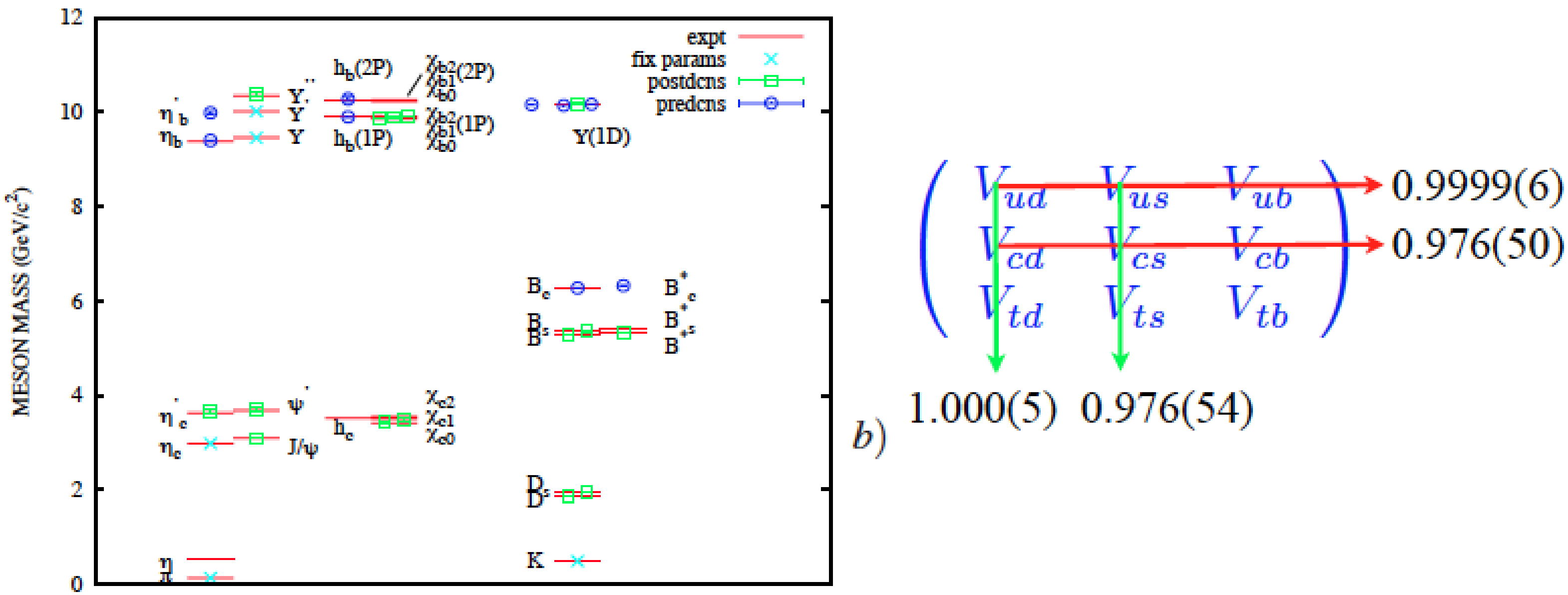
- **Lattice Gauge Theory**:



Nonperturbative QCD methods

- **Lattice Gauge Theory**:

a)



b)

Figure 8: a) The spectrum of gold-plated mesons from lattice QCD, updating [68] to include the prediction of D-wave Upsilon states [51]. b) Tests of unitarity of rows and columns of the CKM matrix now possible using lattice QCD results, except for V_{ud} from nuclear β decay [38]. I have taken direct determinations of V_{us}, V_{cd} and V_{cs} from semileptonic modes [35, 39, 43], although there is no particular reason to prefer those over leptonic modes. V_{ts} and V_{td} come from lattice QCD B_s/B_d mixing calculations [38, 57] and V_{cb} from $B \to D^*$ decays $(39.7(1.0) \times 10^{-3}$ [69]) but these are too small to have much impact on row/column unitarity. Unitarity triangle tests, from orthogonality of columns 1 and 3, are discussed in [20].