* Exercise Lec 5: Using the algorithm ID3, compute (manually) the decision tree corresponding to the task of realising simple Boolean formulas (AND, OR, XOR, ...) in 2 variables. Hint: Consider the truth table of the formulas as your training sample.

1) Not:

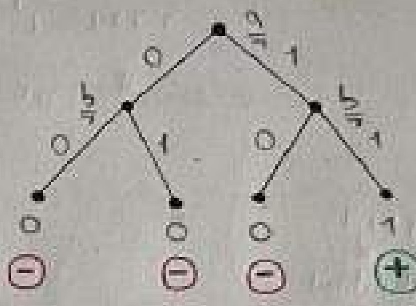
$\overline{A}$

variable 1 (a)	output
0	1
1	0

2) AND:

AB

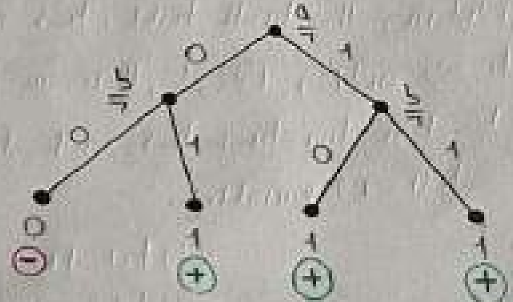
variable 1 (a)	variable 2 (b)	output
0	0	0
0	1	0
1	0	0
1	1	1



3) OR:

$A+B$

variable 1 (a)	variable 2 (b)	output
0	0	0
0	1	1
1	0	1
1	1	1



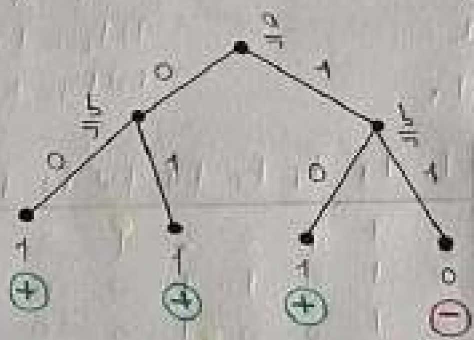
4) NAND:

Not + and gates:

$\overline{AB}$

variable 1 (a)	variable 2 (b)	output
0	0	1
0	1	1
1	0	1
1	1	0

$\begin{cases} 0 & \text{if both are } 1 \\ 1 & \text{otherwise} \end{cases}$



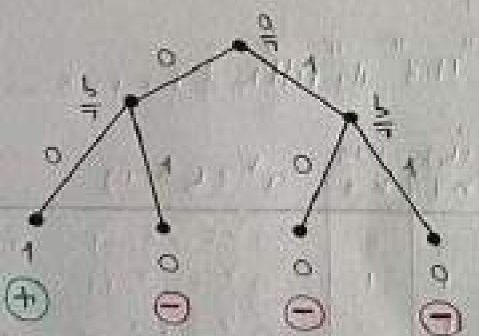
5) NOR:

Not + or gates:

$\overline{A+B}$

variable 1 (a)	variable 2 (b)	output
0	0	1
0	1	0
1	0	0
1	1	0

$\begin{cases} 1 & \text{if both are } 0 \\ 0 & \text{otherwise} \end{cases}$



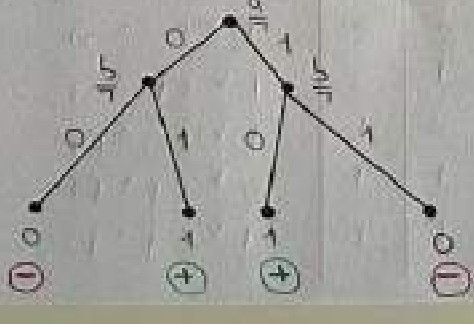
6) XOR:

Exclusive or gates:

$A \oplus B$

variable 1 (a)	variable 2 (b)	output
0	0	0
0	1	1
1	0	1
1	1	0

$\begin{cases} 0 & \text{if both are } 0 \text{ or } 1 \\ 1 & \text{otherwise} \end{cases}$

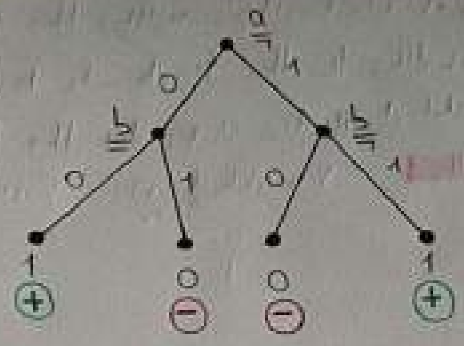


7) XNOR =

exclusive nor gates

$\overline{A \oplus B}$

variable 1 (a)	variable 2 (b)	output
1	1	1
1	0	0
0	1	0
0	0	1



$$\begin{cases} 1 & \text{both 0 or 1} \\ 0 & \text{otherwise} \end{cases}$$

a	b
0	1
0	0
1	1
1	0

$\underbrace{\quad}_{0 \text{ or } 1}$ $\underbrace{\quad}_{0 \text{ or } 1}$

2 possibilities $\rightarrow 2 \times 2 = 2^2 = 2^n = 4 \rightarrow n$ = the number of variables = n variables

13) Considering the truth table, for a n like $n=2$, there are $2^2 = 4$ sequences of length 2 made up of 0's and/or 1's. So, there are 2^n sequences of length n made up of 0's and/or 1's. To make a boolean function for each of these sequences ($2^2 = 4$), there are $(2)^{2^n} = 2^4 = 16$ boolean functions of ($n=2$) variables.

1) $F_0 = 0$
2) $F_1 = 1$
3) $F_2 = a$
4) $F_3 = b$
5) $F_4 = !a$
6) $F_5 = !b$
7) $F_6 = a \wedge b$ (and)
8) $F_7 = a \vee b$ (or)
9) $F_8 = !a \wedge b$
10) $F_9 = !a \vee b$
11) $F_{10} = a \wedge !b$
12) $F_{11} = a \vee !b$
13) F_{12} = a nor b
14) F_{13} = a xnor b
15) F_{14} = a xor b
16) F_{15} = a nand b

$$(2)^{2^n} = 2^{2^2} = 16$$

a	b	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
1	0	0	1	1	0	0	1	0	1	0	0	1	1	0	0	1	1
0	1	0	1	0	1	1	0	0 or	1	1	1	0	0	0	0	1	1
0	0	0	1	0	0	1	0	0	0	1	1	0	1	1	1	0	1
1	1	0	1	1	1	0	1	1	1	1	1	0	1	0	1	0	0

Now ID3 algorithm = For $n=3 \Rightarrow 2^3 = 8 \Rightarrow a \vee (b \wedge c) \Rightarrow$

(a) x_1	(b) x_2	(c) x_3	$a \vee (b \wedge c)$
0	0	0	$0 \vee (0 \wedge 0) \rightarrow 0$ −
0	0	1	$0 \vee (0 \wedge 1) \rightarrow 0$ −
0	1	0	$0 \vee (1 \wedge 0) \rightarrow 0$ −
1	0	0	$1 \vee (0 \wedge 0) \rightarrow 1$ +
1	1	0	$1 \vee (1 \wedge 0) \rightarrow 1$ +
1	0	1	$1 \vee (0 \wedge 1) \rightarrow 1$ +
0	1	1	$0 \vee (1 \wedge 1) \rightarrow 1$ +
1	1	1	$1 \vee (1 \wedge 1) \rightarrow 1$ +

$$P_0 = P_- = \frac{3}{3+5} = 3/8 = P_{False}$$

$$P_1 = P_+ = \frac{5}{3+5} = 5/8 = P_{True}$$

$$E(S) = -[P_- \log(P_-) + P_+ \log(P_+)]$$

$$G(S,a) = E(S) - \sum_{v \in v(a)} \frac{|S_{a=v}|}{|S|} E(S_{a=v})$$

$$E(S) = -[3/8 \log 3/8 + 5/8 \log 5/8] = 0.93$$

$$S = 3 + 5 = 8$$

$$|S_{x1=1}| = 4 \longrightarrow P_+ = 4/4 = 1 , P_- = 0/4 = 0 \longrightarrow E(S_{x1=1}) = -[0 \times \log(0) + 1 \times \log(1)] = -\log^1 = 0$$

$$|S_{x1=0}| = 4 \longrightarrow P_+ = 1/4 , P_- = 3/4 \longrightarrow E(S_{x1=0}) = -[3/4 \log(3/4) + 1/4 \log(1/4)] = 0.80$$

$$\Rightarrow G(S,x1) = E(S) - [4/8 \times E(S_{x1=1}) + 4/8 \times E(S_{x1=0})] = [0.93] - [(4/8) \times 0 + (4/8) \times (0.80)]$$

$$= 0.53$$

$$|S_{x2=1}| = 4 \longrightarrow P_+ = 3/4 , P_- = 1/4 \longrightarrow E(S_{x2=1}) = -[1/4 \log(1/4) + 3/4 \log(3/4)] = 0.80$$

$$|S_{x2=0}| = 4 \longrightarrow P_+ = 2/4 , P_- = 2/4 \longrightarrow E(S_{x2=0}) = -[2/4 \log(2/4) + 2/4 \log(2/4)] = 1$$

$$\Rightarrow G(S,x2) = E(S) - [4/8 \times E(S_{x2=1}) + 4/8 \times E(S_{x2=0})] = [0.93] - [1/2 (0.80) + 1/2 (1)] =$$

$$= 0.03$$

$$|S_{x3=1}| = 4 \longrightarrow P_+ = 3/4 , P_- = 1/4 \longrightarrow E(S_{x2=1}) = -[1/4 \log(1/4) + 3/4 \log(3/4)] = 0.80$$

$$|S_{x3=0}| = 4 \longrightarrow P_+ = 2/4 , P_- = 2/4 \longrightarrow E(S_{x2=0}) = -[2/4 \log(2/4) + 2/4 \log(2/4)] = 1$$

$$\Rightarrow G(S,x3) = E(S) - [4/8\, E(S_{x3=1}) + 4/8\, E(S_{x3=0})] = [0.93] - [1/2 (0.80) + 1/2 (1)] =$$

$$= 0.03$$