* the proof of $E(x)$ of Poisson distribution function=

x = a discrete random variable with Poisson distribution function.

we have Poisson distribution with parameter λ.

the variance is λ. $\Rightarrow Var(x) = \lambda$

by considering that the variance = [Expectation of square] - [square of expectation]

$\Rightarrow Var(x) = E(x^2) - (E(x))^2$

if $f(x) = x^2 \Rightarrow E(f(x)) = E(x^2) = \sum f(x) P(X=x) = \sum x^2 P(X=x) \Rightarrow$

so from expectation of function of discrete random variable, we have $\Rightarrow$

$$E(x^2) = \sum_{\text{all possible } x} x^2 P(X=x)$$

so by considering the definition of Poisson distribution function, we have=

$$E(x^2) = \sum_{k\geq 0} k^2 \frac{1}{k!} \lambda^k e^{-\lambda} = \lambda e^{-\lambda} \sum_{k\geq 0} k^2 \times \frac{1}{k!} \times \lambda^{(k-1)} = \lambda e^{-\lambda} \sum_{k\geq 1} k^2 \times \frac{1}{k(k-1)!} \times \lambda^{(k-1)}$$

$$= \lambda e^{-\lambda} \sum_{k\geq 1} \frac{k}{(k-1)!} \lambda^{k-1} = \lambda e^{-\lambda} \left[\sum_{k\geq 1} (k-1) \frac{1}{(k-1)!} \lambda^{k-1} + \sum_{k\geq 1} \frac{1}{(k-1)!} \lambda^{k-1} \right] =$$

$k = 0$

$k-1 = 0$

$k \geq 1$ (change of limits)

$$\lambda e^{-\lambda} \left[\sum_{k\geq 2} \lambda \frac{1}{(k-2)!} \lambda^{k-2} + \sum_{k\geq 1} \frac{\lambda^{k-1}}{(k-1)!} \right] =$$

$k-1=0 \rightarrow$ change of limits

$k-2=0 \rightarrow k=2 \Rightarrow k\geq 2$

if $i = k-2$

$j = k-1$

$$\Rightarrow \lambda e^{-\lambda} \left[\lambda \underbrace{\sum_{i\geq 0} \frac{1}{i!} \lambda^i}_{e^{\lambda}} + \underbrace{\sum_{j\geq 0} \frac{1}{j!} \lambda^j}_{e^{\lambda}} \right] =$$

$k\geq 2 = i+2 \geq 2 \Rightarrow i \geq 0$

$k \geq 1 = j+1 \geq 1 \Rightarrow j \geq 0$

by considering Taylor series Expansions of exponential functions =

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < +\infty$$

$$\lambda e^{-\lambda} \left[\lambda e^{\lambda} + e^{\lambda} \right] = \lambda^2 + \lambda \Rightarrow E(x^2) = \lambda^2 + \lambda \Rightarrow Var(x) = E(x^2) - (E(x))^2 \Rightarrow$$

$$\lambda = \lambda^2 + \lambda + (E(x))^2 \Rightarrow E(x) = \lambda$$