Show that $\sigma'(y) = \sigma(y)(1-\sigma(y))$

Solution.

We know $\sigma(y) = \dfrac{1}{1+e^{-y}}$

By using reciprocal notable derivative.

$$\left(1/f(x)\right)' = -\frac{f'(x)}{f(x)^2}$$

$$\Rightarrow \quad \sigma(y)' = \left(\frac{1}{1+e^{-y}}\right)' = -\frac{(1+e^{-y})'}{(1+e^{-y})^2}$$

Applying derivative as the sum of all the derivatives of numerator.

$$\sigma'(y) = -\frac{(1+e^{-y})'}{(1+e^{-y})^2} = \frac{e^{-y}}{(1+e^{-y})^2} = \frac{e^{-y}}{(1+e^{-y})(1+e^{-y})}$$

$$= \left(\frac{e^{-y}}{1+e^{-y}}\right)\left(\frac{1}{1+e^{-y}}\right) = \left(\frac{1}{1+e^{-y}}\right)\left(\frac{e^{-y}}{1+e^{-y}}\right)$$

By adding and subtracting 1 from second term of the product.

$$\sigma'(y) = \frac{1}{1+e^{-y}}\left(\frac{e^{-y}}{1+e^{-y}}\right) = \frac{1}{1+e^{-y}}\left(1 + \frac{e^{-y}}{1+e^{-y}} - 1\right)$$

$$= \frac{1}{1+e^{-y}}\left(1 + \frac{e^{-y} - 1 - e^{-y}}{1+e^{-y}}\right)$$

$$= \frac{1}{1+e^{-y}}\left(1 + \frac{-1}{1+e^{-y}}\right) = 1/1+e^{-y}\left(1 - \frac{1}{1+e^{-y}}\right)$$

By replacing $\sigma(y) = \frac{1}{1+e^{-y}}$, we get,

$$\sigma'(y) = \frac{1}{1+e^{-y}}\left(1 - \frac{1}{1+e^{-y}}\right) = \sigma(y)\left(1-\sigma(y)\right).$$

Hence proved.

$\square$