Compute

$$\lim_{n\to\infty} n^{\alpha}\left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$$

One has

$$\sin x = x - \frac{x^3}{6} + o(x^3), \qquad \text{as } x \to 0,$$

so that (setting $x = \frac{1}{n}$)

$$n^{\alpha}\left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right) = n^{\alpha}\left(\frac{1}{n} - \frac{1}{n} + \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right) = \frac{1}{6n^{3-\alpha}} + o\left(\frac{1}{n^{3-\alpha}}\right)$$

Hence

$$\lim_{n\to\infty} n^{\alpha}\left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{1}{6n^{3-\alpha}} + o\left(\frac{1}{n^{3-\alpha}}\right) = \lim_{n\to\infty} \frac{1}{6n^{3-\alpha}}\left(1 + o\left(6\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{6n^{3-\alpha}} \cdot \lim_{n\to\infty} \left(1 + o\left(6\right)\right) = \lim_{n\to\infty} \frac{1}{6n^{3-\alpha}} = \lim_{n\to\infty} \frac{1}{6n^{3-\alpha}} = \begin{cases} 0 & \forall \alpha < 3 \\ \frac{1}{6} & \alpha = 3 \\ +\infty & \forall \alpha > 3 \end{cases}$$

If someone has doubts about the fact that

$$\lim_{n\to\infty} \left(o\left(6\right)\right) = 0$$

(actually this is true for every non-zero constant, not only for 6), she/he should recall the notion of "little o":

Definition *A function* $\phi(n)$ *is* "a little o" *of* $\psi(n)$ *—and one writes* $\phi(n) = o(\psi(n))$ *for* $n \to \infty$*— if*

$$\lim_{n\to\infty} \frac{\phi(n)}{\psi(n)} = 0.$$

Therefore, if $\psi(n)$ is a non-zero constant (as our 6 above), i.e. $\psi(n) \equiv k \in \mathbb{R}\backslash\{0\}$, one has

$$\lim_{n\to\infty} \frac{\phi(n)}{k} = 0,$$

which is obviously equivalent to

$$\lim_{n\to\infty} \phi(n) = 0.$$