Given a function \( f : A \rightarrow B \)

- \( f \) is called injective if \( a_1, a_2 \in A \) and \( a_1 \neq a_2 \) implies \( f(a_1) \neq f(a_2) \)

- Example:

  \[
  f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^3
  \]

  \( f(a_1) = f(a_2) \) implies \( a_1 = a_2 \) (star)
Is it injective?

Apply (*)

\[ f(x_1) = f(x_2) \]

\[ x_1^2 + 3 = x_2^2 + 3 \]

\[ x_1^2 = x_2^2 \]

\[ x_1 = x_2 \]

(since \( x_1, 0 < x_2, 0 \) \( \implies x_1 = x_2 \)

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\[ f : A \rightarrow B \]

is injective if

\[ \forall a, b \in A \exists \exists a \in A \text{ s.t. } f(a) = b \]

\[ f(A) = \{ f(a) \mid a \in A \} = B \]
(in general we have)

$\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Remark: $E, F$

$E \cap F$

$E \subseteq F$ and $F \subseteq E$

Example: $f: \mathbb{R}^+ \rightarrow \mathbb{R}$

$f(x) = 3 + x^2$ subject to

No: $-1 \in \mathbb{R}$

$-1 = 3 + x$

$x^2 = -4$ not possible

$1 \in \mathbb{R}$

$1 = 3 + x$

$x^2 = 2$ possible
If we change the codomain $\mathbb{R}$ with $B = \{y \in \mathbb{R} \mid y > 3\}$

$g: \mathbb{R}^+ \to B$ is

surjective $\Rightarrow g(x) = f(x)$

Proof: Let $y \in B$

$y = g(x) = 3 + x^2$

$x^2 = 3 - y \geq 0$

$x = \sqrt{3 - y}$

This $g: \mathbb{R}^+ \to B$ is both injective and surjective

so $g$ is bijective
Definition: If \( f : A \to B \)
both inj. and surj. we say \( f \) is bijective.

If \( f : A \to B \) is bijective we say that there is a one to one correspondence between \( A \) and \( B \).

\[
\begin{align*}
&\{ I_1, I_2, \ldots, I_n \} \\
&\{ \{ 1 \}, \{ 2, 3 \}, \{ 1, 2, 3 \}, \ldots, B \}
\end{align*}
\]
Def A is finite if it can be put in one-to-one correspondence with some $\mathbb{N}_n$, for some $n \geq 1$.

I say that “A has n elements”.

Def A is infinite if it is not finite.

Examples: $\mathbb{N}$ is infinite.

$\mathbb{Q}_{0}^{+}$: positive rational numbers

$f: \mathbb{N} \rightarrow \mathbb{Q}_{0}^{+}$,

$n \mapsto \begin{cases} 0 & \text{if } n = 0, \\ n & \text{if } n \geq 1. \end{cases}$

$f$ is injective,

$2n + 1 = 2n_{2} + 1$ 
$n_{2} = 2n_{2}$
Surjective?

\[ y \in O^+ \quad y = 2n + 1 \text{ for some } n. \]

\[ \Rightarrow f \text{ is bijective} \]

**Definition**: A set is said countable if either it is finite or it is in 1-1 relationship with \( \mathbb{N} \).

\[ \mathbb{Z} = \{ n \in \mathbb{N} \mid n \leq 0 \} \]

\[ -3 -2 -1 0 1 2 3 \]

**Exercise**: Prove that \( \mathbb{Z} \) is countable.
\( Q = \{ \text{rational numbers} \} = \{ \text{fractions} \} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \} \\
\Rightarrow Q \supseteq \mathbb{N} = \{ \frac{0}{1}, \frac{1}{1}, \frac{2}{1} \} \\
\mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \\
-2 -1 \phantom{.} 0 \phantom{.} 1 \phantom{.} 2 \phantom{.} 3 \\

\text{Theorem: } Q \text{ is countable} \\
i.e. \exists f \\
f: \mathbb{N} \rightarrow Q \text{ bijective} \\
(\text{idea: } Z \times Z \rightarrow \mathbb{R})
\[ A \times B = \begin{cases} (a, b) & a \in A, \ b \in B \\
, \ & \\
, \ Z, \ Q \]

**Pythagoras' Theorem:**

\[ c^2 = a^2 + b^2 \]

**Example:**

\[ a = 1, \ b = 1 \]

\[ c = \sqrt{2} \]

\[ c^2 = 2 \]
Theorem: There exist no rational numbers $c$ such that $c^2 = 2$.

Proof: By contradiction.

\[ \sqrt{2} \notin \mathbb{Q} \]

\[ \sqrt{2} \in \mathbb{Q} \]

\[ \sqrt{2} = \frac{m}{n} \]

Assume that \( \frac{m}{n} \) is already reduced in order that $m$ and $n$ have no common factors.
\[
2 = \frac{m^2}{n^2}
\]

\[m^2 = 2n^2 \Rightarrow m^2 \text{ is even} \]

\[\Rightarrow m \text{ is even} \quad (\text{by exercise})\]

\[m^2 = 2q \quad q \in \mathbb{N}\]

\[4q^2 = 2n^2 \]

\[n^2 = 2q^2 \]

\[n^2 \text{ is even} \]

\[n \text{ is even} \]

q.e.d.
How to enlarge $Q$?

$N \rightarrow Z \rightarrow Q$

$m \in Q \quad r \in Q$

$s \in Q$

\[
\frac{m}{n} + \frac{r}{s} = \frac{ms + rn}{ns}
\]

\[
\frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns}
\]

\[
\frac{m}{n} \div \frac{r}{s} = \frac{m}{n} \cdot \frac{s}{r}
\]

\[
\frac{m}{n} \div \frac{r}{s}
\]

$\bar{n} Q$

$q_1 = \frac{m_1}{n_1}$

$q_2 = \frac{m_2}{n_2}$

$q_1 \leq q_2$

$a \text{ der}$
Definition: \( A \subseteq \mathbb{Q} \) is a lower bound for \( A \) if \( 1 \in \mathbb{Q} \) such that \( 1 \leq a \) for all \( a \in A \).

Example: \( A_1 = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \)

\[ A_2 = \{ \frac{5}{14}, \frac{11}{5}, \frac{3}{2} \} \]

a lower bound for \( A_1 \) is \( \frac{1}{28} \neq \frac{1}{24} \)

0 is a lower bound

-3 is a lower bound

The "best" lower bound is 0.
Indeed if I take, \( R > 0 \) \( \beta > 0 \) \( \beta > 0 \) I can find \( p \in A_{1} \) \( 0 < q \leq \beta \) \( \frac{1}{4} \leq \frac{1}{2} \) \( \frac{1}{2} \leq 1 \) \( 0 \) is the only one satisfying \( \odot \) Definition the "best lower bound" is a lower bound that satisfies \( \odot \) We call the best lower bound the [INFINITY] of \( A_{1} \)