Lesson 2 - 23/09/2022

Notations / definitions for the study of vector fields.

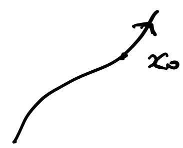
We often suppose that $X \in \mathcal{C}^\infty(\Omega; \mathbb{R}^n)$.

- X is called COMPLETE if solutions are defined for every $t \in \mathbb{R}$.

- $(t, x_0) \mapsto \underbrace{x(t; 0, x_0)}_{= \varphi^t(x_0)}$ $\to$ the solution of $\begin{cases} \dot{x} = X(x) \\ x(0) = x_0 \end{cases}$ at time t

  is the **flow** of the v.f.

- The image of $t \mapsto x(t; 0, x_0)$ is the **orbit** (or **trajectory**) of the v.f. passing through x_0.



- The set of all obits = phase portrait.

- A point $\bar{x} \in \Omega$ is called **equilibrium** if $X(\bar{x}) = 0$.

  Ω = phase-space.

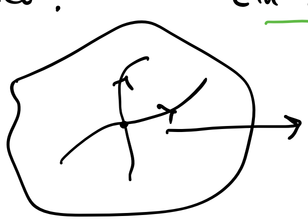
- Important properties of a flow.

$$\varphi^0(x_0) = x_0 \qquad \forall x_0 \in \Omega$$

$$\varphi^{t+s}(x_0) = \varphi^t(\varphi^s(x_0)) = \varphi^s(\varphi^t(x_0)) \qquad \forall t, s \in \mathbb{R}$$

$$\boxed{\Rightarrow} \quad \underbrace{\varphi^t \circ \varphi^{-t}}_{} \underset{\text{semigroup property}}{=} \varphi^{t-t} \underset{\text{1st property}}{=} \varphi^0 = \underbrace{\mathrm{id}}_{} \Rightarrow \varphi^{-t} = (\varphi^t)^{-1}$$

For at least Lip. continuous flows, orbits cannot intersect! (in finite time!!)



$\to$ the corresponding Cauchy problem admits two different solutions ↯.

- Two first examples of 2-dim v.f.

$\boxed{1}$ On $\mathbb{R}^2$, let consider $\quad (x = (x_1, x_2))$

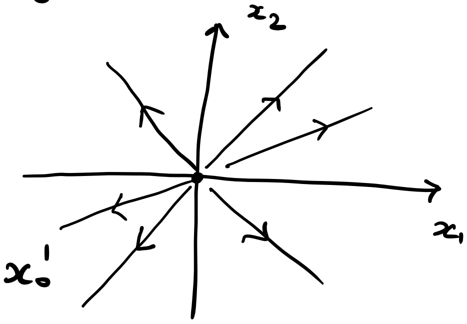
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (\text{Briefly } \dot{x} = Ax)$$

(linear vector field on $\mathbb{R}^2$)

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2 \end{cases} \longrightarrow \begin{cases} x_1(t) = e^t x_0^1 \\ x_2(t) = e^t x_0^2 \end{cases} \longrightarrow$$

Phase portrait

$$x(t) = e^t \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$$



$\boxed{2}$ On $\mathbb{R}^2$

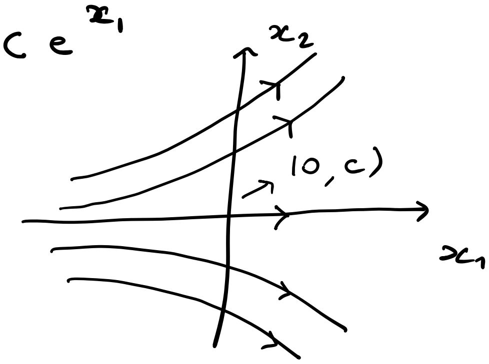
$$\begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = x_2 \end{cases} \longrightarrow \begin{cases} x_1(t) = t + x_0^1 \\ x_2(t) = e^t x_0^2 \end{cases}$$

$\longrightarrow$ eliminate t

$$\begin{cases} t = x_1 - x_0^1 \\ x_2 = e^{x_1 - x_0^1} x_0^2 = \underbrace{x_0^2 e^{-x_0^1}}_{c} e^{x_1} \end{cases}$$

c = constant dep. on initial position!

$$\Rightarrow \quad x_2 = c\, e^{x_1}$$

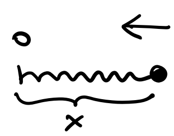


(the phase space is foliated by these curves!)

Some examples from mechanics

$\boxed{1}$ Harmonic oscillator (spring)

$$\ddot{x} = -\omega^2 x \qquad (m\ddot{x} = -Kx)$$



$(\omega^2 = k/m)$

$k>0$ = elastic constant.

As a first order v.f.

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x \end{cases} \Rightarrow \quad X\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$$

We solve equation : $\ddot{x} + \omega^2 x = 0$

$\lambda^2 + \omega^2 = 0 \Leftrightarrow \lambda = \pm i\omega$

$\Rightarrow \quad x(t; \underbrace{A, B}_{\text{dep. on initial conditions}}) = A\cos(\omega t) + B\sin(\omega t) \qquad (A, B \in \mathbb{R})$

$\Rightarrow \dot{x}(t; A, B) = -A\omega \sin(\omega t) + B\omega\cos(\omega t)$

we impose initial conditions : $x(0; A, B) = A = x_0$

$$\underset{\dot{x}}{\overset{v}{\shortparallel}}(0; A, B) = B\omega = v_0$$

$$\Rightarrow \begin{cases} A = x_0 \\ B = v_0/\omega \end{cases}$$

$$\begin{cases} x(t) = x_0\cos(\omega t) + v_0/\omega \sin(\omega t) \\ v(t) = \dot{x}(t) = -x_0\omega\sin(\omega t) + v_0\cos(\omega t) \end{cases}$$

Eliminate t ....

$$[x(t)]^2 = x_0^2\cos^2(\omega t) + \frac{v_0^2}{\omega^2}\sin^2(\omega t) + 2\ldots$$

$$[v(t)]^2 = x_0^2\omega^2\sin^2(\omega t) + v_0^2\cos^2(\omega t) - 2\ldots$$

$$\boxed{[x(t)]^2 + \frac{1}{\omega^2}[v(t)]^2 = x_0^2 + \frac{v_0^2}{\omega^2}}$$

constant > 0

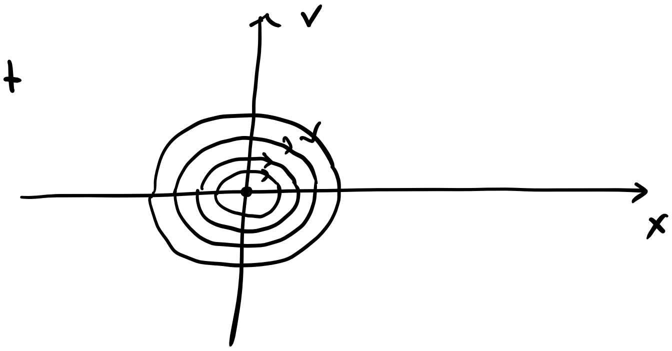
(= 0 only for $x_0 = v_0 = 0$)

$\Downarrow$ ORBITS satisfies eq

$$X^2 + \frac{1}{\omega^2} Y^2 = \text{const} \quad (>0)$$

eq. of an ellipse!

Phase - portrait



2 Gravitational v.f.

$$\ddot{x} = g \quad \Rightarrow \quad \begin{cases} \dot{x} = v \\ \dot{v} = g \end{cases} \quad \Rightarrow \quad \begin{cases} x(t) = x_0 + v_0 t + \frac{1}{2} g t^2 \\ v(t) = v_0 + g t \end{cases}$$

Eliminate the time t. $\quad t = \dfrac{v - v_0}{g}$

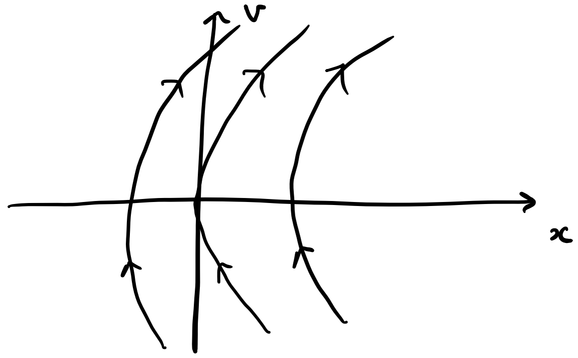
$$\Rightarrow x = x_0 + v_0 \left(\frac{v - v_0}{g} \right) + \frac{1}{2} g \left(\frac{v - v_0}{g} \right)^2 =$$

$$= x_0 + \cancel{\frac{v_0 v}{g}} \underbrace{- \frac{v_0^2}{g}} + \frac{1}{2} g \frac{v^2}{g^2} + \underbrace{\frac{1}{2} g \frac{v_0^2}{g^2}} \cancel{- \frac{1}{2} g \frac{(v v_0)}{g^2}}$$

$$= x_0 - \frac{1}{2} \frac{v_0^2}{g} + \frac{1}{2} \frac{v^2}{g}$$

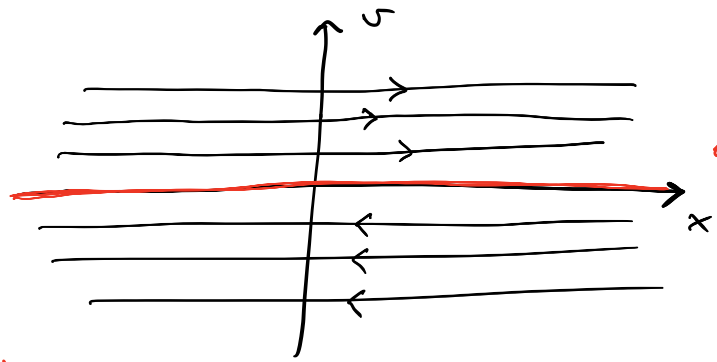
Hence: $\quad x = \underbrace{x_0 - \dfrac{v_0^2}{2g}}_{\text{const. dep. on initial conditions}} + \dfrac{1}{2} \dfrac{v^2}{g}$

$\Rightarrow$ The phase-phase is foliated by parabolas.



3 Free particle (No forces!)

$$\ddot{x} = 0 \rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = 0 \end{cases} \Rightarrow \begin{cases} x(t) = x_0 + v_0 t \\ v(t) \equiv v_0 \end{cases}$$



4 Harmonic repeller

$$\ddot{x} = \underbrace{\omega^2}_{>0} x \rightarrow \ddot{x} - \omega^2 x = 0 \rightarrow \lambda^2 - \omega^2 = 0 \rightarrow \lambda = \pm\omega$$

$$x(t; A, B) = A e^{\omega t} + B e^{-\omega t}$$

$$v(t; A, B) = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

$A, B \in \mathbb{R}$ dep. on initial conditions.

$$\begin{cases} \dot{x} = v \\ \dot{v} = \omega^2 x \end{cases} \rightarrow \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

(the first order v.f.)

We impose initial data $x(0) = x_0$ and $v(0) = v_0$ and we obtain ....

$$\begin{cases} A = \dfrac{x_0}{2} + \dfrac{v_0}{2\omega} \\[2ex] B = \dfrac{x_0}{2} - \dfrac{v_0}{2\omega} \end{cases}$$

Therefore :

$$\begin{cases} x(t; 0, x_0, v_0) = \left(\dfrac{x_0}{2} + \dfrac{v_0}{2\omega}\right) e^{\omega t} + \left(\dfrac{x_0}{2} - \dfrac{v_0}{2\omega}\right) e^{-\omega t} \\[2ex] v(t; 0, x_0, v_0) = \omega \left(\dfrac{x_0}{2} + \dfrac{v_0}{2\omega}\right) e^{\omega t} - \omega \left(\dfrac{x_0}{2} - \dfrac{v_0}{2\omega}\right) e^{-\omega t} \end{cases}$$

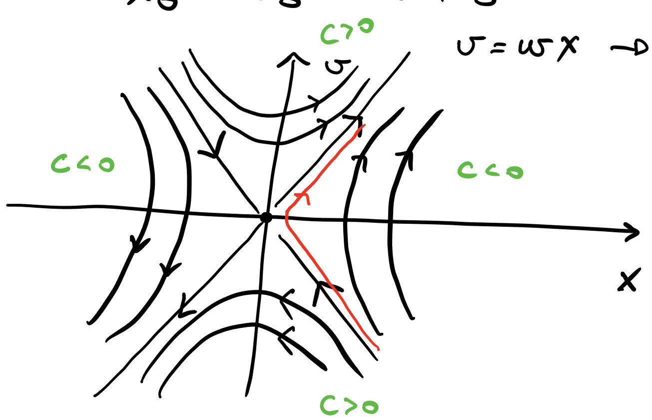
Remarks

- $\exists !$ equilibrium $(0, 0)$
- If $x_0 - \dfrac{v_0}{\omega} = 0 \iff \underline{v_0 = x_0 \omega} \Rightarrow$

$$\begin{cases} x(t) = x_0 e^{\omega t} \\ v(t) = \underbrace{\omega x_0}_{= v_0} e^{\omega t} = v_0 e^{\omega t} \end{cases} \Rightarrow$$

$$\frac{x}{x_0} = \frac{v}{v_0} = \frac{v}{\omega x_0} \Rightarrow v = \omega x$$

$v = \omega x \rightarrow$ invariant curve for the dynamics.



- Same fact for the invariant curve $v = -\omega x$

Other curves on the phase-plane ?!

By eliminating the time, with the same calculations for the harmonic oscillator, we prove that ORBITS satisfy

$$v^2 - \omega^2 x^2 = \underbrace{v_0^2 - \omega^2 x_0^2}_{= c \text{ const. dep. on initial data.}}$$

↙

iperbole

Ex

- Determine equilibria, type of equilibria and phase portrait of

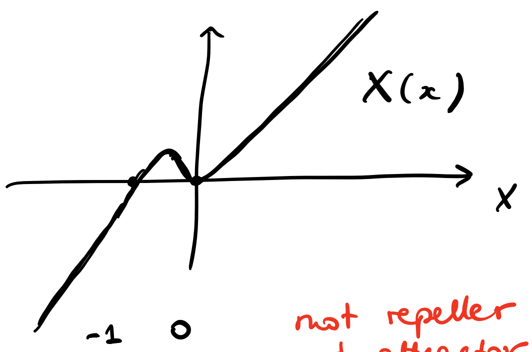
$\dot x = x^4 + x = x(x^3 + 1)$

$\dot x = x^3 + x^2 = x^2(x + 1)$

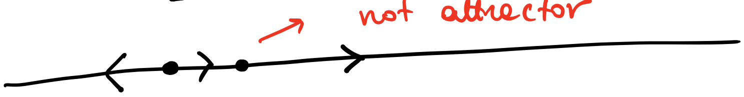
$\dot x = \sin x$

$\dot x = X(x) = x^2(x+1)$

EQUILIBRIA? $X(x) = 0 \iff x^2(x+1) = 0$ → $x = -1$, $x = 0$



not repeller not attractor (unstable)



repeller (unstable)

- Determine equilibria of

$\ddot x = 1 - x$ $\qquad$ $\ddot x = (1 - x^2)(x + \dot x)$

$\ddot x = -x - \dot x$

$$\begin{cases} \dot{x} = \underline{v} \\ \dot{v} = \underline{-x - v} \end{cases}$$ at first order.

$$X(x, v) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

$$X(x, v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} v = 0 \\ -x - v = 0 \end{cases}$$

$\exists!$ equilibrium : the origin !

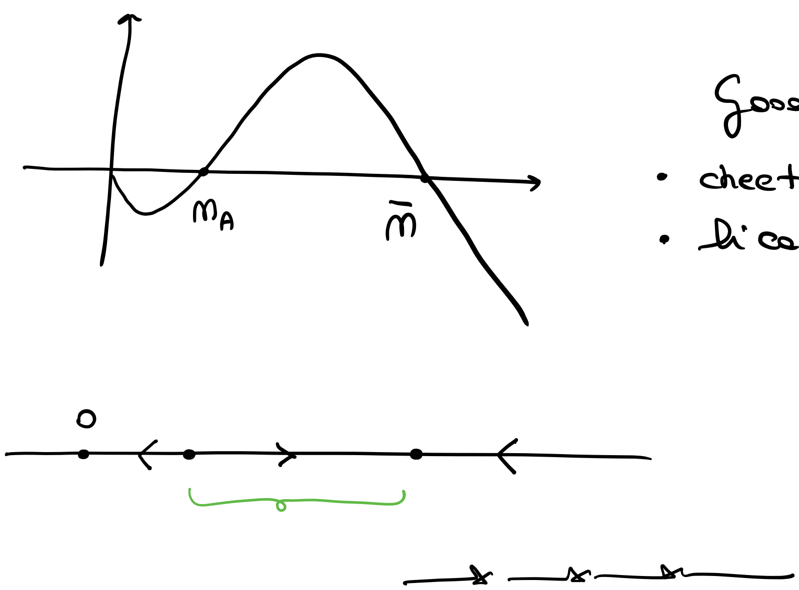
On monday

We start from a correction of the logistic model

$$\dot{m} = K\left(1 - \frac{m}{\bar{m}}\right)\underbrace{\left(\frac{m}{m_A} - 1\right)}_{}m \qquad m_A > 0$$

logistic model

where $0 < m_A < \bar{m}$



Good model for

- cheetah
- bisoni