Lesson 1 - 28/09/2022 -

- Dynamical systems.
- Lagrangian mechanics.
- A bit of Calculus of Variations.

Dynamics

- mid - 1600s, Newton → diff. equations
  ↘ 2- bodies problem (earth, sun).

- Can Newton's analytical methods be applied to the 3-bodies problem? (earth, sun, moon ...)
  → essentially impossible to solve.

  - The breakthrough came with Poincaré in the late 1800s. He introduced a qualitative rather that quantitative point of view.

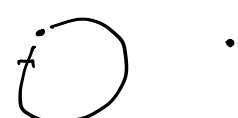
Ex

Quantitative questions

- Exact position of a planet?
- Trajectories of planets?

Qualitative questions

- Is the solar system stable forever?
- Is the solar system stable under small perturbation?
- Are there special trajectories? (periodic, fixed ....)



Finally, Poincaré understood the possibility of CHAOS.

- To understand this "revolution", we start by some simple examples from population dynamics.

• Two examples in population dynamics.

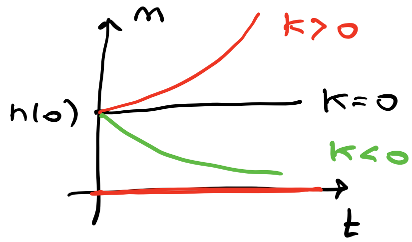
① Malthusian growth model.

$m(t)$ = population size at time t.

$K \in \mathbb{R}$

$\dot{m}(t) = K\, m(t)$ → the speed of growth is proportional to the population size.

$$m(t) = n(0)\, e^{Kt}$$



② Verhulst (logistic) growth model.

We introduce a logistic correction: the speed of growth decreases when the population size increase.

$$\dot{m}(t) = K\left(\frac{\bar{m} - m(t)}{\bar{m}}\right) m(t)$$

$\bar{m}$ = carrying capacity.

- if $\bar{m} - m(t) > 0 \iff m(t) < \bar{m} \Rightarrow$ the population increases.
- if $\bar{m} - m(t) < 0 \iff m(t) > \bar{m} \Rightarrow$ the population decreases.

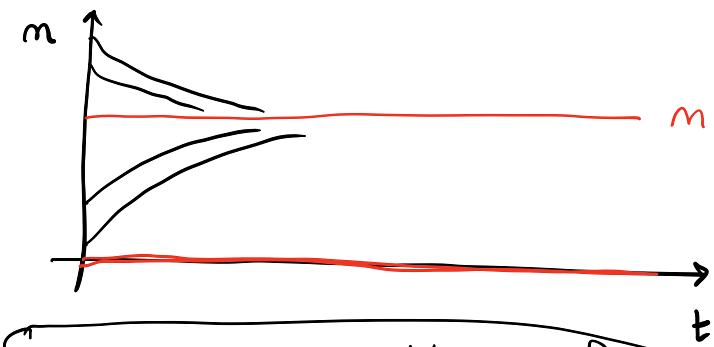
$g = K/\bar{m}$ → $\dot{x} = Kx - gx^2$ → By separation of variables...

(.... more calculation that the previous case $\dot{x} = Kx$)

$$m(t) = \frac{K\, m(0)}{g\, n(0) + (K - g\, n(0))\, e^{-Kt}} \qquad (g = K/\bar{m})$$

Qualitative properties of the solution (explicit, in such a case)

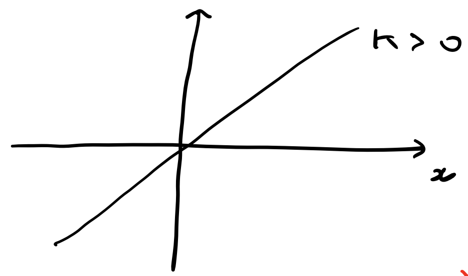
1) If $m(0) = 0$ then $m(t) \equiv 0$
2) if $m(0) = \frac{K}{g}$ then $m(t) \equiv K/g$
3) $\lim_{t \to +\infty} m(t) = K/g \quad \forall\, m(0) > 0$!!



Question: Can we obtain infos on dynamics without solving eqs ?! YES !!

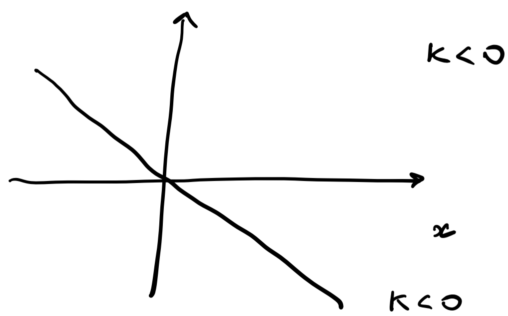
① **Malthusian model**

$$\dot{x} = \underbrace{Kx = X(x)}$$

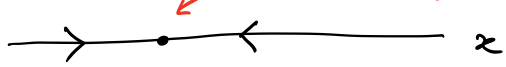


EQUILIBRIUM <=> $X(x) = 0$

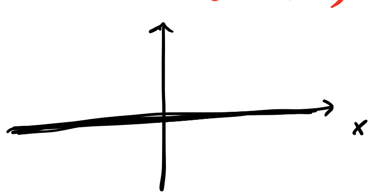
(repeller)



EQUILIBRIUM <=> $X(x) = 0$



(attractor)



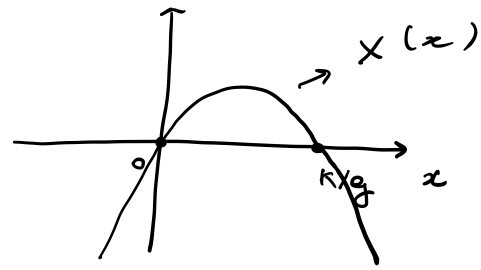
All fixed points.

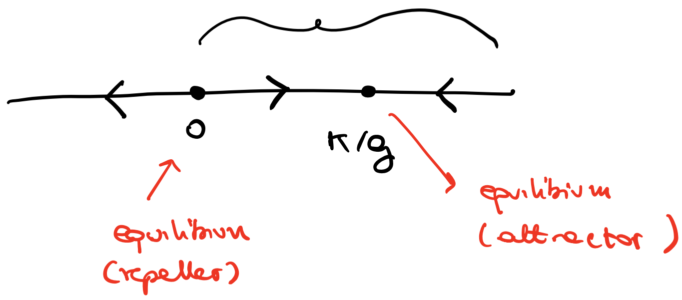
↳ EVERY POINT IS A (STABLE) EQUILIBRIUM.

(2) Verhulst model $\quad \underbrace{\dot{x} = kx - gx^2}_{= X(x)} \qquad (k, g > 0)$

$$X(x) = \overbrace{x(k - gx)}^{} = 0 \iff \begin{cases} x = 0 \\ \text{or} \\ x = k/g \end{cases}$$

$$X(x) = x(k - gx) > 0 \iff 0 < x < k/g$$





- Other two examples of 1-dim. vector fields.

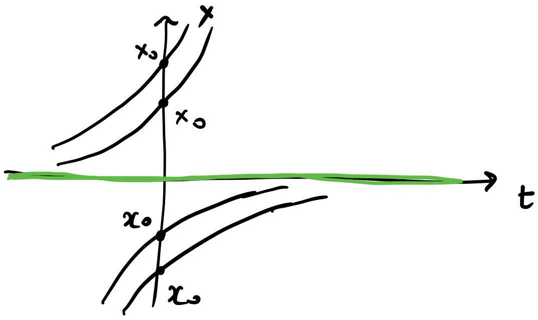
3) $\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases}$

- $\dfrac{dx}{dt} = x^2 \longrightarrow t = \displaystyle\int_{x_0}^{x} \frac{dy}{y^2} = -\frac{1}{y}\Big|_{x_0}^{x} = -\frac{1}{x} + \frac{1}{x_0}$

$$\Rightarrow t + c = -\frac{1}{x} \Rightarrow x(t) = \frac{-1}{t + c}$$

Imposing that $x(0) = x_0$, we obtain that $c = -\dfrac{1}{x_0}$

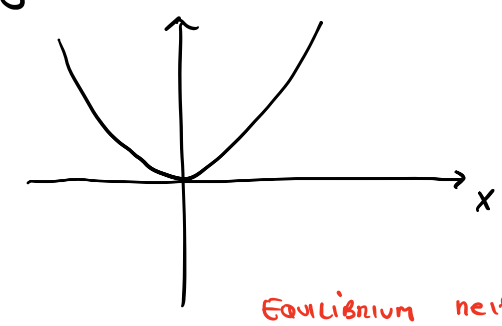
$$\Rightarrow x(t; \underbrace{0, x_0}_{\text{in. cond.}}) = \frac{-1}{t - \frac{1}{x_0}} = \frac{x_0}{1 - x_0 t}$$

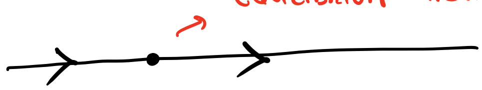


$x_0 > 0$: $\lim_{t \to \frac{1}{x_0}^-} x(t) = +\infty$ | We go to $+\infty$ in finite time

$x_0 < 0$: $\lim_{t \to +\infty} x(t) = 0$ | we go to 0 when $t \to +\infty$

Qualitative study $\dot{x} = X(x) = x^2$





Equilibrium neither repeller nor attractor!!

4) On $\mathbb{R}^+ = (0, +\infty)$, let consider

$$\begin{cases} \dot{x} = -1/x \\ x(0) = x_0 \end{cases} \qquad \frac{dx}{dt} = -1/x \Rightarrow -t = \frac{1}{2}x^2 - \frac{1}{2}x_0^2$$

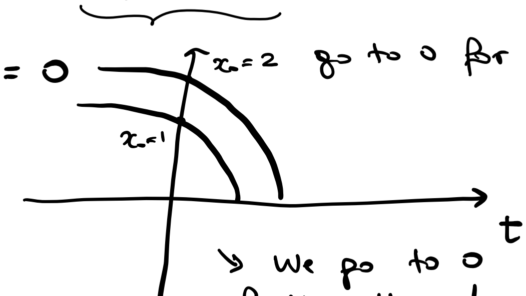
$$\Rightarrow -t + c = \frac{1}{2}x^2$$

$$\Rightarrow [x(t)]^2 = -2t + c \rightarrow x(0) = x_0$$

we obtain that $c = x_0^2$

$$\Rightarrow x(t) = x(t;\underbrace{0, x_0}_{\text{in. cond.}}) = \sqrt{c - 2t} = \sqrt{\underbrace{x_0^2 - 2t}}$$

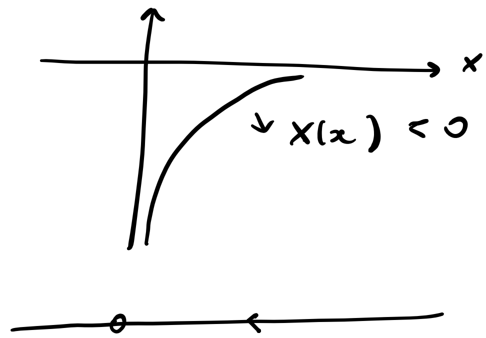
And $\lim_{t \to \frac{x_0^2}{2}^-} x(t) = 0$



go to 0 for $t \to 1^-$

↳ We go to 0 in finite time!

$$\dot{x} = X(x) = -\frac{1}{x}$$



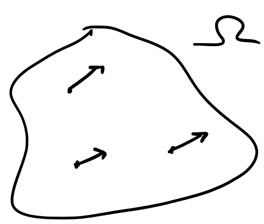
$X(x) < 0$ always

Some recalls on vector fields

$\Omega \subseteq \mathbb{R}^n$ open set $\quad (\Omega = \mathbb{R}^n$ eventually$)$.

A vector field of class C^k $(k \geq 0$ or $\infty)$ on Ω is a map

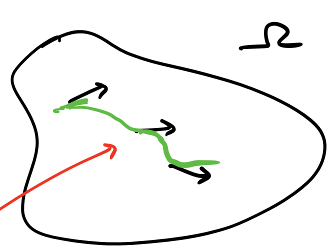
$$X : \Omega \longrightarrow \mathbb{R}^n, \quad x \longmapsto X(x) \quad \text{of class } C^k$$



To every point x I assign an "arrow" $X(x)$.

A diff. equation on Ω is $\dot{x} = X(x)$

Solving a diff. eq. means finding curves whose derivatives are the arrows previously assigned by the v.f.



A solution of $\dot{x} = X(x)$ on Ω is a C^1 function
$x : I \to \Omega \quad (I \subseteq \mathbb{R}$ time-interval$)$ such that

$$\boxed{\dot{x}(t) = X(x(t)) \quad \forall\, t \in I}$$

$$\begin{cases} \dot{x} = X(x) \\ x(0) = x_0 \end{cases} \longrightarrow \underline{\text{Cauchy problem}} \quad (*)$$

What about existence / uniqueness of solutions of Cauchy problems? ANSWER:

Cauchy existence & uniqueness theorem

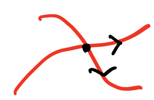
Let $X : \Omega \longrightarrow \mathbb{R}^n$ Lipschitz-continuous, that is:

$\exists \ \lambda > 0$ such that

$$|X(x_2) - X(x_1)| \leq \lambda \, |x_1 - x_2| \quad \forall \, x_1, x_2 \in \Omega.$$

Then

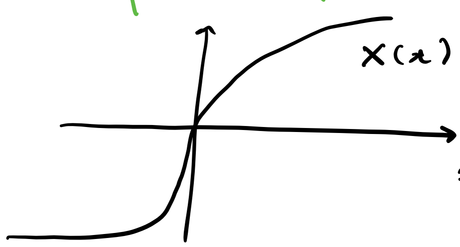
1) Every Cauchy problem $(*)$ has solution.
2) The solution is unique.

(trajectories CANNOT intersect!! →  NO!)

(Recall that $X \in C^1 \Rightarrow X \in Lip$, we usually have $X \in C^\infty$ .... !!)

- EXAMPLE of NO-uniqueness ($X \in C^0$, not Lip)

$$\begin{cases} \dot{x} = x^{1/3} \\ x(0) = 0 \end{cases}$$

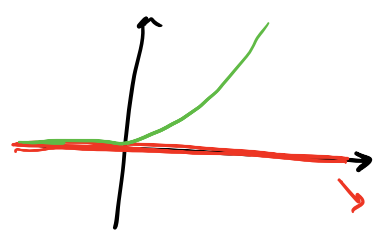


$X(x) = x^{1/2}$ not globally Lip.

The previous Cauchy problem has 2 solutions.

$x_1(t) \equiv 0$

$$x_2(t) = \begin{cases} \left(\frac{2}{3} t\right)^{3/2} & t > 0 \\ 0 & t \leq 0 \end{cases}$$



$\to x_1(t)$

$$\dot{x}_2(t) = \frac{\cancel{2}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \left(\frac{2}{3}t\right)^{1/2} = \underbrace{\left[\left(\frac{2}{3}t\right)^{3/2}\right]}_{x_2(t) \ (t>0)}{}^{1/3} = \left(x_2(t)\right)^{1/3} =$$

$$= X(x_2(t))$$

- **Remark**

It holds that, if $X \in \mathcal{C}^\infty$ then $x \in \mathcal{C}^\infty$. <u>Proof</u>

In fact, by hyp. $x \in \mathcal{C}^1$. Let consider

$$\underset{\mathcal{C}^\infty}{\underset{\cap}{X}} \circ \underset{\in \mathcal{C}^1}{x} = \underbrace{X(x)}_{\in \mathcal{C}^1} = \underbrace{\dot{x}}_{\in \mathcal{C}^2} \Rightarrow x \in \mathcal{C}^2.$$

Repeat the argument, with $x \in \mathcal{C}^2$.

$$\underset{\mathcal{C}^\infty}{\underset{\cap}{X}} \circ \underset{\in \mathcal{C}^2}{x} = X(x) = \underset{\in \mathcal{C}^2}{\dot{x}} \Rightarrow x \in \mathcal{C}^3$$

and so on .... $x \in \mathcal{C}^\infty$ as the v.f. X !!

- what can we deduce if $X \in \mathcal{C}^2(X, \mathbb{R}^n)$ or

$$X \in \mathcal{C}^k(X, \mathbb{R}^m) \ ??$$

— x — x —