

# Linear Programming and the Simplex method

Luigi De Giovanni

Dipartimento di Matematica, Università di Padova

# Mathematical Programming models

$$\begin{aligned} \min(\max) \quad & f(x) \\ \text{s.t.} \quad & g_i(x) = b_i \quad (i = 1 \dots k) \\ & g_i(x) \leq b_i \quad (i = k + 1 \dots k') \\ & g_i(x) \geq b_i \quad (i = k' + 1 \dots m) \\ & x \in \mathbb{R}^n \end{aligned}$$

- $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  is a vector (column) of  $n$  **REAL** variables;
- $f$  e  $g_i$  are functions  $\mathbb{R}^n \rightarrow \mathbb{R}$
- $b_i \in \mathbb{R}$

# Linear Programming (LP) models

$f$  e  $g_i$  are **linear** functions of  $x$

$$\begin{array}{ll} \min(\max) & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots k) \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \quad (i = k + 1 \dots k') \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \quad (i = k' + 1 \dots m) \\ & x_j \in \mathbb{R} \quad (i = 1 \dots n) \end{array}$$

Notice: for the moment, just **CONTINUOUS variables are considered!!!**

We need different methods for models with integer or binary variables.

# Resolution of an LP model

- *Feasible solution*:  $x \in \mathbb{R}^n$  satisfying all the constraints
- *Feasible region*: set of all the feasible solutions  $x$
- *Optimal solution*  $x^*$  [min]:  $c^T x^* \leq c^T x, \forall x \in \mathbb{R}^n, x$  feasible.

Solving a LP model is determining if it:

- is unfeasible
- is unlimited
- has a (finite) optimal solution

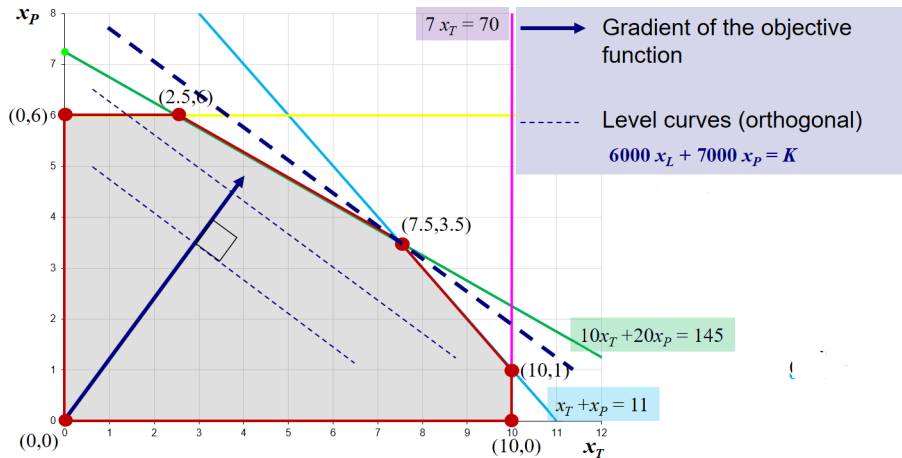
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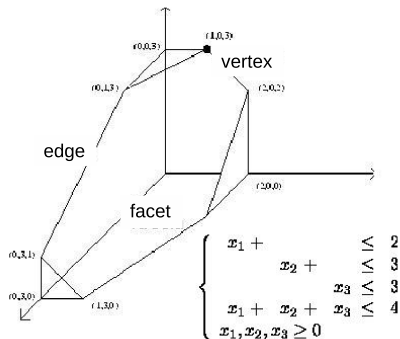
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# Solving an LP: example



# Geometry of LP

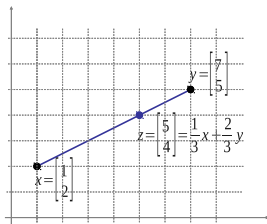
The feasible region is a **polyhedron** (intersection of a finite number of closed half-spaces and hyperplanes in  $\mathbb{R}^n$ )



LP problem:  $\min(\max)\{c^T x : x \in P\}$ ,  $P$  is a polyhedron in  $\mathbb{R}^n$ .

# Vertex of a polyhedron: definition

- $z \in \mathbb{R}^n$  is a **convex combination** of two points  $x$  and  $y$  if  $\exists \lambda \in [0, 1]$  :  
$$z = \lambda x + (1 - \lambda)y$$

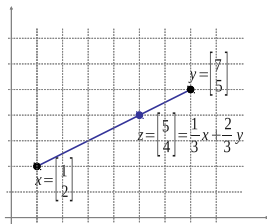


- $z \in \mathbb{R}^n$  is a **strict convex combination** of two points  $x$  and  $y$  if  $\exists \lambda \in (0, 1)$  :  $z = \lambda x + (1 - \lambda)y$ .
- $v \in P$  is **vertex of a polyhedron**  $P$  if it is **not a strict convex combination** of two *distinct* points of  $P$ :  
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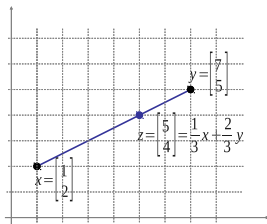
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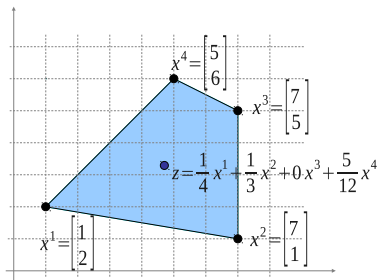


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# Representation of polyhedra

$z \in \mathbb{R}^n$  is **convex combination** of  $x^1, x^2 \dots x^k$  if  $\exists \lambda_1, \lambda_2 \dots \lambda_k \geq 0$  :

$$\sum_{i=1}^k \lambda_i = 1 \text{ and } z = \sum_{i=1}^k \lambda_i x^i$$



Theorem: representation of polyhedra [Minkowski-Weyl] - case 'limited'

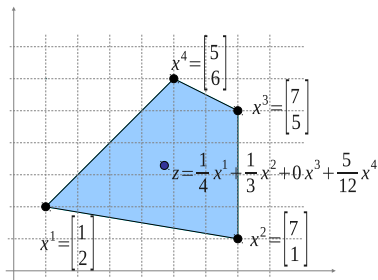
Polydron *limited*  $P \subseteq \mathbb{R}^n$ ,  $v^1, v^2, \dots, v^k$  ( $v^i \in \mathbb{R}^n$ ) vertices of  $P$

if  $x \in P$  then  $x = \sum_{i=1}^k \lambda_i v^i$  with  $\lambda_i \geq 0, \forall i = 1..k$  and  $\sum_{i=1}^k \lambda_i = 1$   
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# Optimal vertex: from graphical intuition to proof

## Theorem: optimal vertex (fix *min* objective function)

LP problem  $\min\{c^T x : x \in P\}$ ,  $P$  non empty and limited

- LP ha optimal solution
- **one of the optimal solution of LP is a vertex of  $P$**

Proof:

$$V = \{v^1, v^2 \dots v^k\} \quad v^* = \arg \min_{v \in V} c^T v$$

$$c^T x = c^T \sum_{i=1}^k \lambda_i v^i = \sum_{i=1}^k \lambda_i c^T v^i \geq \sum_{i=1}^k \lambda_i c^T v^* = c^T v^* \sum_{i=1}^k \lambda_i = c^T v^*$$

Summarizing:  $\forall x \in P, c^T v^* \leq c^T x$  ■

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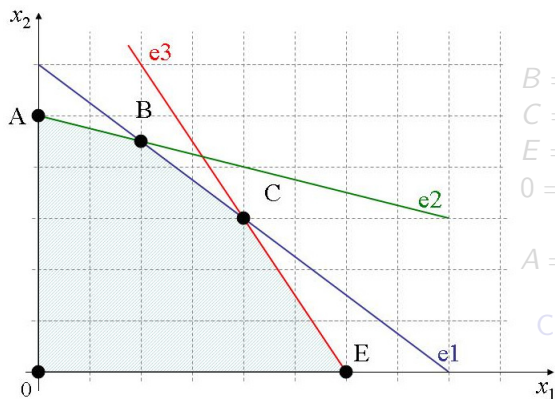
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# Vertex comes from intersection of generating hyperplanes

$$\begin{aligned} \max \quad & 13x_1 + 10x_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 \leq 24 \quad (\text{e1}) \\ & x_1 + 4x_2 \leq 20 \quad (\text{e2}) \\ & 3x_1 + 2x_2 \leq 18 \quad (\text{e3}) \\ & x_1, x_2 \geq 0 \end{aligned}$$



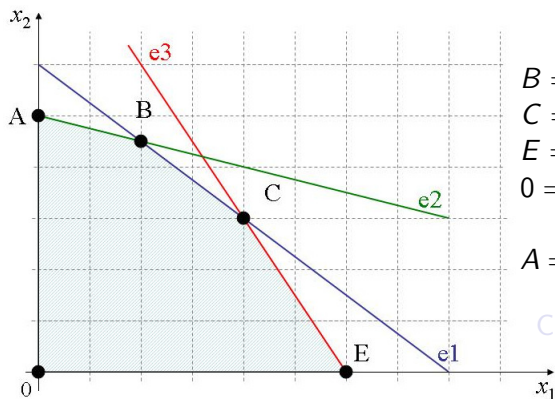
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$C = e1 \cap e3$	$(4, 3)$	82
$E = e3 \cap (x_2 = 0)$	$(6, 0)$	78
$0 = (x_1 = 0) \cap (x_2 = 0)$	$(0, 0)$	0
$A = e2 \cap (x_1 = 0)$	$(0, 5)$	50

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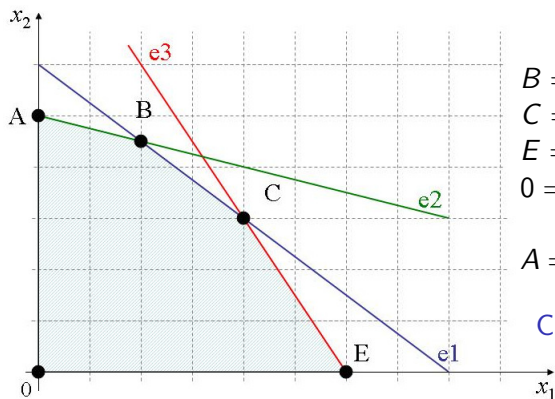


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# Algebraic representation of vertices

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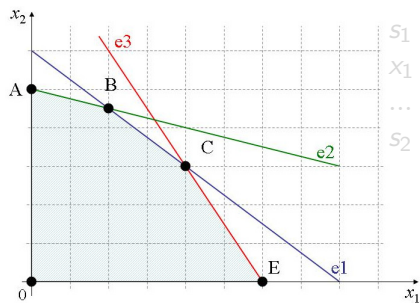
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5 - 3 = 2 degrees of freedom:

we can set (any) two variables to 0 and obtain a unique solution!



$$s_1 = s_2 = 0 \quad (2, 9/2, 0, 0, 3) \quad B$$

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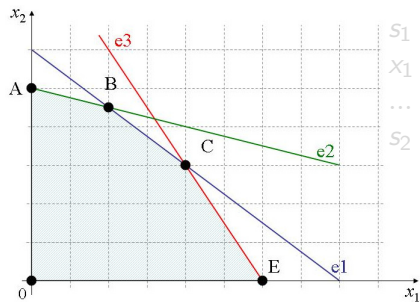
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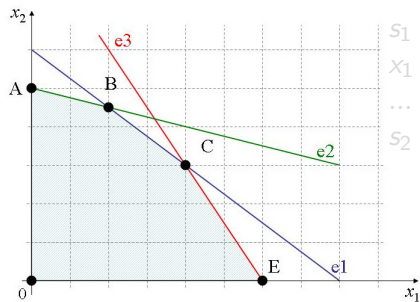
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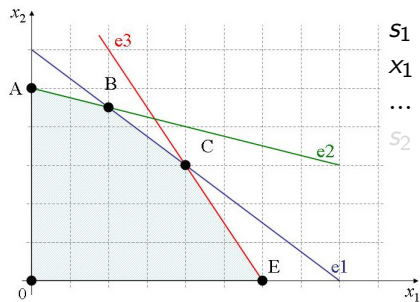
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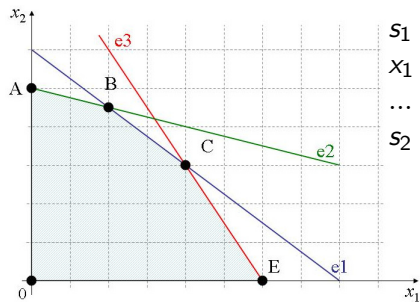
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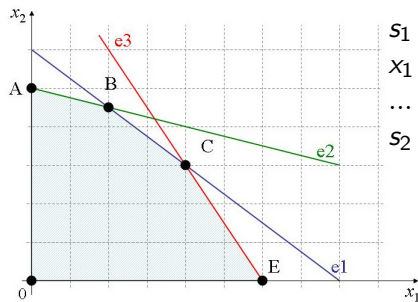
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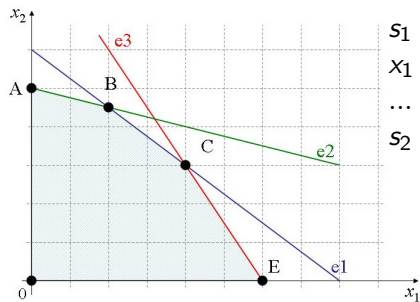
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## Standard form for LP problems

$$\begin{array}{ll} \min & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m) \\ & x_i \in \mathbb{R}_+ \quad (i = 1 \dots n) \end{array}$$

- **minimizing** objective function (if not, multiply by  $-1$ );
- variables  $\geq 0$ ; (if not, substitution)
- all constraints are equalities; ( $+/-$  slack/surplus variables)
- $b_i \geq 0$ . (if not, multiply by  $-1$ )

## Standard form: example

$$\begin{aligned} \max \quad & 5(-3x_1 + 5x_2 - 7x_3) + 34 \\ \text{s.t.} \quad & -2x_1 + 7x_2 + 6x_3 - 2x_1 \leq 5 \\ & -3x_1 + x_3 + 12 \geq 13 \\ & x_1 + x_2 \leq -2 \\ & x_1 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \hat{x}_1 &= -x_1 & (\hat{x}_1 \geq 0) \\ x_3 &= x_3^+ - x_3^- & (x_3^+ \geq 0, x_3^- \geq 0) \end{aligned}$$

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# Linear algebra: definitions

- column vector  $v \in \mathbb{R}^{n \times 1}$ :  $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$
- row vector  $v^T \in \mathbb{R}^{1 \times n}$ :  $v^T = [v_1, v_2, \dots, v_n]$
- matrix  $A \in \mathbb{R}^{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- $v, w \in \mathbb{R}^n$ , scalar product  $v \cdot w = \sum_{i=1}^n v_i w_i = v^T w = w^T v$
- Rank of  $A \in \mathbb{R}^{m \times n}$ ,  $\rho(A)$ , max linearly independent rows/columns
- $B \in \mathbb{R}^{m \times m}$  invertible  $\iff \rho(B) = m \iff \det(B) \neq 0$

# Systems of linear equations

- *Systems of equations in matrix form*: a system of  $m$  equations in  $n$  variables can be written as:

$$Ax = b, \text{ where } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \text{ e } x \in \mathbb{R}^n.$$

- *Theorem of Rouché-Capelli*:

$$Ax = b \text{ has solutions } \iff \rho(A) = \rho(A|b) = r \text{ (}\infty^{n-r} \text{ solutions)}.$$

- *Elementary row operations*:

- ▶ swap row  $i$  and row  $j$ ;
- ▶ multiply row  $i$  by a non-zero scalar;
- ▶ substitute row  $i$  by row  $i$  plus  $\alpha$  times row  $j$  ( $\alpha \in \mathbb{R}$ ).

Elementary operations on (augmented) matrix  $[A|b]$  leave the same solutions as  $Ax = b$ .

- *Gauss-Jordan method* for solving  $Ax = b$ : make elementary row operations on  $[A|b]$  so that  $A$  contains an identity matrix of dimension  $\rho(A) = \rho(A|b)$ .

# Basic solutions

- **Assumptions:** system  $Ax = b$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $\rho(A) = m$ ,  $m < n$
- **Basis of  $A$ :** square submatrix with maximum rank,  $B \in \mathbb{R}^{m \times m}$
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- imposing  $x_N = 0$ , we obtain a so called **basic solution**:  
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## Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

$$x_j \in \mathbb{R}_+$$

$$\min \quad c^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

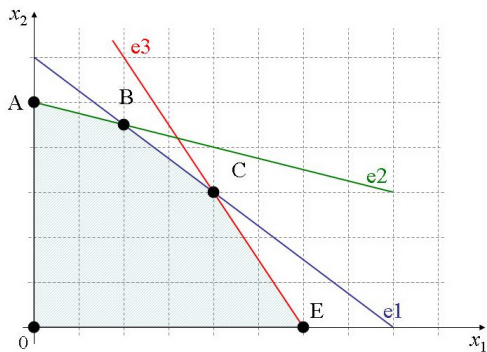
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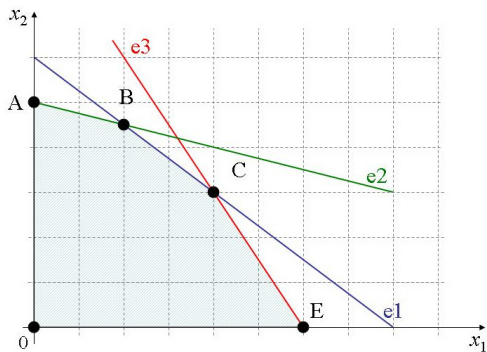
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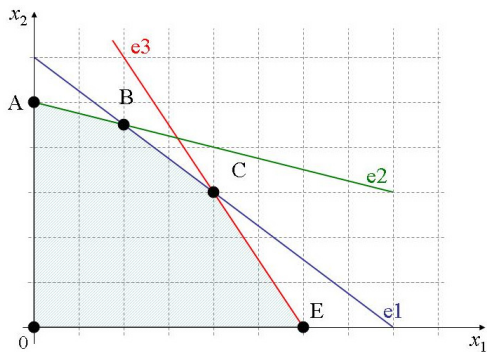
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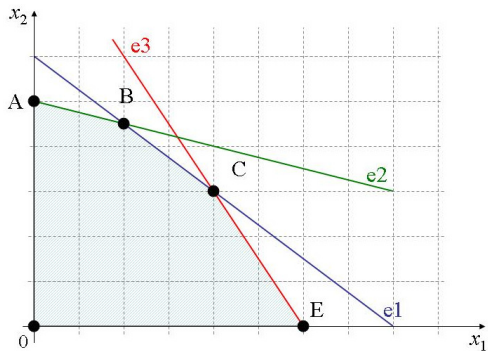
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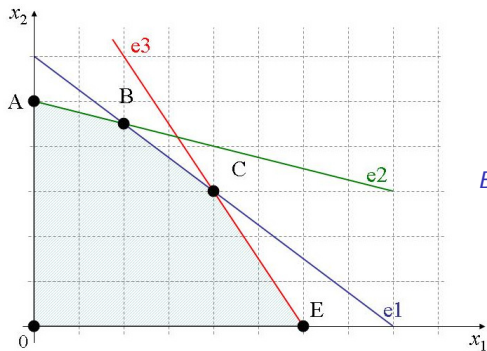
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$$B_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

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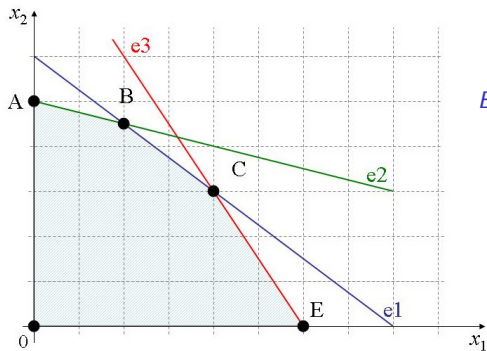
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$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = B_1^{-1}b = \begin{bmatrix} 2 \\ 4,5 \\ 3 \end{bmatrix}$$

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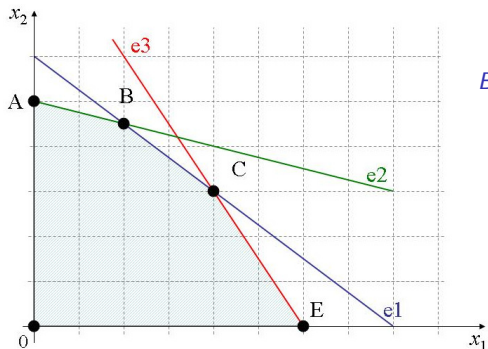
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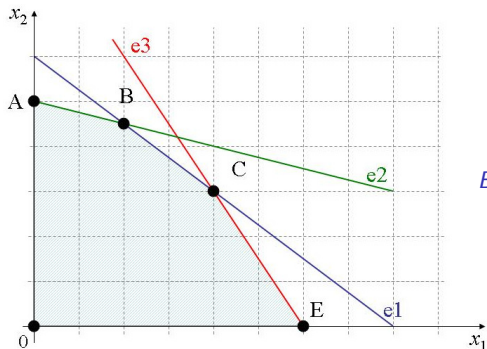
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$$B_2 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

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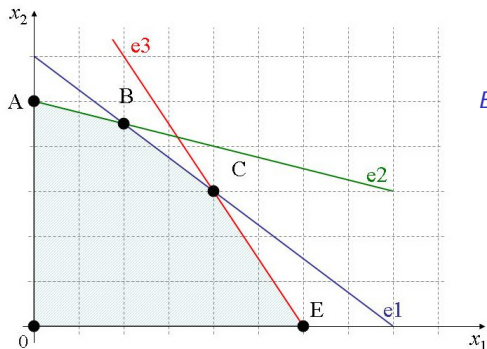
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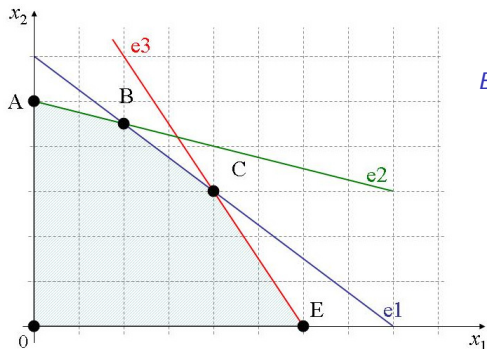
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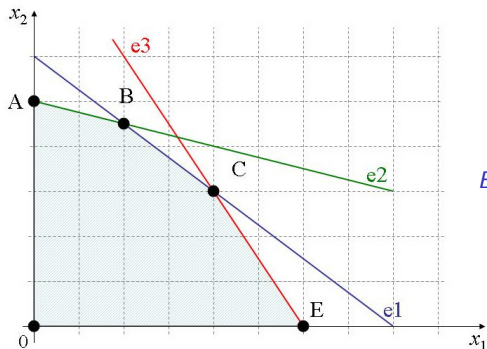
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$$B_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

## Basic solutions and LP in standard form

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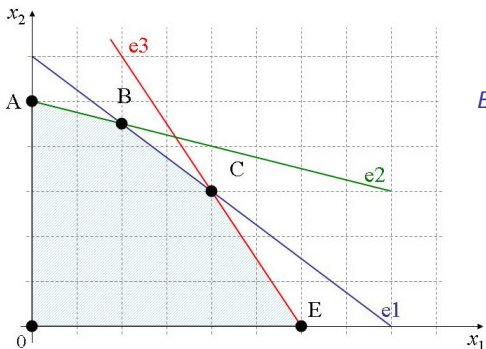
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$$B_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ s_1 \\ s_2 \end{bmatrix} = B_3^{-1}b = \begin{bmatrix} 6 \\ 6 \\ 14 \end{bmatrix}$$

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## Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

$$x_j \in \mathbb{R}_+$$

$$(i = 1 \dots n)$$

$$\min \quad c^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

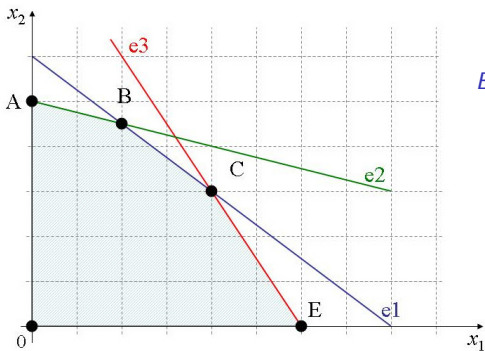
- basis  $B$  gives a **feasible basic solution** if  $x_B = B^{-1}b \geq 0$

$$3x_1 + 4x_2 + s_1 = 24$$

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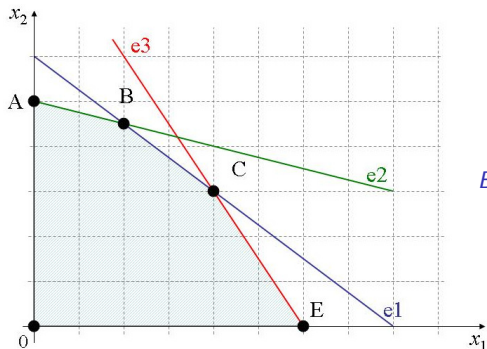
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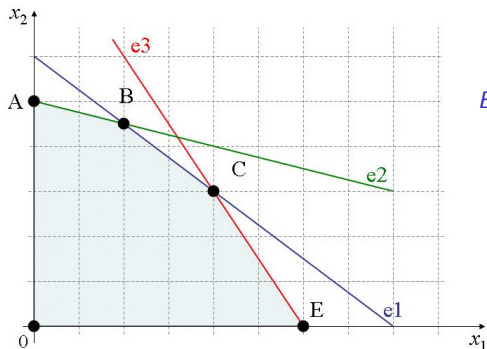
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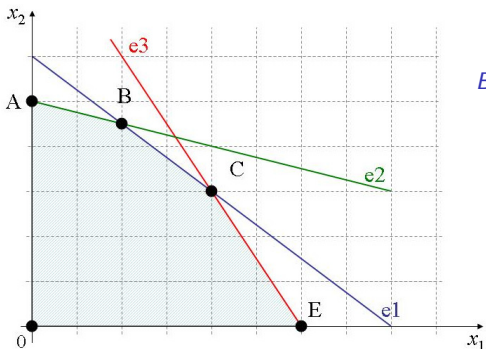
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$$x^T = (18/5 \ 21/5 \ -18/5 \ 0 \ 0) \rightarrow \text{n.f.}!$$

# Vertices and basic solution

Feasible basic solution  $\rightsquigarrow$   $n - m$  variables are 0  $\rightsquigarrow$   
intersection of the right number of hyperplanes  $\rightsquigarrow$  vertex!

$$\text{PL } \min\{c^T x : Ax = b, x \geq 0\} \quad P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

**Theorem: vertices correspond to feasible basic solutions**  
*(algebraic characterization of the vertices of a polyhedron)*

$x$  feasible basic solution of  $Ax = b$   $\iff$   $x$  is a vertex of  $P$

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## Algorithm for LP (case limited): sketch

Consider **all** the feasible basic solutions:

- 1 put the LP in standard form  $\min\{c^T x : Ax = b, x \geq 0\}$
- 2 *incumbent* =  $+\infty$
- 3 **repeat**
- 4 generate a combination of  $m$  columns of  $A$
- 5 let  $B$  be the corresponding submatrix of  $A$
- 6 **if**  $\det(B) == 0$  **then continue** **else** compute  $x_B = B^{-1}b$
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## Example

LP problem in **standard form**:

$$\begin{array}{llllllll} \min & z = & -13x_1 & - & 10x_2 & & & \\ \text{s.t.} & & 3x_1 & + & 4x_2 & + & s_1 & = & 24 \\ & & x_1 & + & 4x_2 & & + & s_2 & = & 20 \\ & & 3x_1 & + & 2x_2 & & & + & s_3 & = & 18 \\ & & x_1 & , & x_2 & , & s_1 & , & s_2 & , & s_3 & \geq & 0 \end{array}$$

an initial **basic feasible solution** (vertex B):

- $B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$      $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- $x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9/2 \\ 3 \end{bmatrix}$      $x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $z_B = c^T x = c_B^T x_B + c_N^T x_N = -71$

## Example

Change basis: **New basic solution**  $\Rightarrow$  one non-basic variable increases  
**affecting**  $x_B$  and  $z_B$

$$\begin{aligned}x_B &= B^{-1}b - B^{-1}N x_N \\z &= c^T x = c_B^T x_B + c_N^T x_N = c_B^T (B^{-1}b - B^{-1}N x_N) + c_N^T x_N \\&= c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N) x_N\end{aligned}$$

Write  $x_B$  and  $z$  as functions of only **non-basic** variables

For the sake of manual computation, use **Gauss-Jordan**:

$$Ax = b \quad \rightsquigarrow \quad [B \ N \mid b] \quad \rightsquigarrow \quad [B^{-1}B = I \ B^{-1}N = \bar{N} \mid B^{-1}b = \bar{b}]$$

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$x_1$	$x_2$	$s_3$	$s_1$	$s_2$	$\bar{b}$
3	4	0	1	0	24
1	4	0	0	1	20
3	2	1	0	0	18

$(R_1/3)$	1	4/3	0	1/3	0	8
$(R_2 - R_1/3)$	0	8/3	0	-1/3	1	12
$(R_3 - R_1)$	0	-2	1	-1	0	-6

$(R_1 - 1/2 R_2)$	1	0	0	1/2	-1/2	2
$(3/8 R_2)$	0	1	0	-1/8	3/8	9/2
$(R_3 + 3/4 R_2)$	0	0	1	-5/4	3/4	3

$$x_1 = 2 - \frac{1}{2} s_1 + \frac{1}{2} s_2$$

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- Equalities have to be always satisfied...:

$$\begin{aligned}x_1 &= 2 + 1/2 s_2 \\x_2 &= 9/2 - 3/8 s_2 \\s_3 &= 3 - 3/4 s_2\end{aligned}$$

- while preserving non-negativity:

$$\begin{aligned}x_1 \geq 0 &\Rightarrow 2 + 1/2 s_2 \geq 0 \Rightarrow s_2 \geq -4 \quad \text{always!} \\x_2 \geq 0 &\Rightarrow 9/2 - 3/8 s_2 \geq 0 \Rightarrow s_2 \leq 12 \\s_3 \geq 0 &\Rightarrow 3 - 3/4 s_2 \geq 0 \Rightarrow s_2 \leq 4\end{aligned}$$

- New **feasible** and **better** solutions with  $s_1 = 0$  and  $0 \leq s_2 \leq 4$
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## Example

New basic solution!  $s_2$  (now  $> 0$ ) takes the place of  $s_3$  (now  $= 0$ ):

$$B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$
$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad z_B = c^T x = c_B^T x_B + c_N^T x_N = -82$$

Same arguments as before:  $x_B$  and  $z$  as a function of  $x_N$ :

$$\begin{aligned} x_1 &= 4 + \frac{1}{3} s_1 - \frac{2}{3} s_3 \\ x_2 &= 3 - \frac{1}{2} s_1 - \frac{1}{2} s_3 \\ s_2 &= 4 + \frac{5}{3} s_1 - \frac{4}{3} s_3 \\ z &= -82 + \frac{2}{3} s_1 + \frac{11}{3} s_3 \end{aligned}$$

**Optimal solution!** Visited 2 out of  $\binom{5}{3} = 10$  possible basis

## Example

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$$B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$
$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad z_B = c^T x = c_B^T x_B + c_N^T x_N = -82$$

Same arguments as before:  $x_B$  and  $z$  as a function of  $x_N$ :

$$\begin{aligned} x_1 &= 4 + 1/3 s_1 - 2/3 s_3 \\ x_2 &= 3 - 1/2 s_1 - 1/2 s_3 \\ s_2 &= 4 + 5/3 s_1 - 4/3 s_3 \\ z &= -82 + 2/3 s_1 + 11/3 s_3 \end{aligned}$$

Optimal solution! Visited 2 out of  $\binom{5}{3} = 10$  possible basis

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## LP in *canonical* form

PL  $\min\{z = c^T x : Ax = b, x \geq 0\}$  is in **canonical form with respect to basis  $B$**  if all basic variables and the objective are explicitly written as functions of **non-basic variables only**:

$$\begin{aligned} z &= \bar{z}_B + \bar{c}_{N_1} x_{N_1} + \bar{c}_{N_2} x_{N_2} + \dots + \bar{c}_{N_{(n-m)}} x_{N_{(n-m)}} \\ x_{B_i} &= \bar{b}_i - \bar{a}_{iN_1} x_{N_1} - \bar{a}_{iN_2} x_{N_2} - \dots - \bar{a}_{iN_{(n-m)}} x_{N_{(n-m)}} \quad (i = 1 \dots m) \end{aligned}$$

$\bar{z}_B$  scalar (objective function value for the corresponding basic solution)

$\bar{b}_i$  scalar (value of basic variable  $i$ )

$B_i$  index of the  $i$ -th basic variable ( $i = 1 \dots m$ )

$N_j$  index of the  $j$ -th non-basic variable ( $j = 1 \dots n - m$ )

$\bar{c}_{N_j}$  coefficient of the  $j$ -th non-basic variable in the objective function (**reduced cost of the variable with respect to basis  $B$** )

$-\bar{a}_{iN_j}$  coefficient of the  $j$ -th non-basic variable in the constraints that makes explicit the  $i$ -th basic variable

## Simplex method: optimality check

- **Reduced cost** of a variable: marginal unit increment of the objective function
- The reduced cost of a basis variable is  $\bar{c}_{B_i} = 0$

### Theorem: Sufficient optimality conditions

Given an LP and a feasible basis  $B$ , if all the reduced costs with respect to  $B$  are  $\geq 0$ , then  $B$  is an optimal basis

$$\bar{c}_j \geq 0, \forall j = 1 \dots n \Rightarrow B \text{ optimal}$$

- Notice: the inverse is not true! [there may be optimal basic solutions with negative reduced costs]

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- From feasible basis  $B$ , obtain a  $\tilde{B}$  **adjacent, feasible, improving**
- **One** column ( $\approx$  variable) enters and one variable leaves the basis

- **Entering** variable (improvement): any  $x_h : \bar{c}_h < 0$

$$z = \bar{z}_B + \bar{c}_h x_h = \bar{z}_{\tilde{B}} \leq \bar{z}_B$$

- **Leaving** variable (feasibility): [min ratio rule]

$$x_{B_i} \geq 0 \Rightarrow b_i - \bar{a}_{ih} x_h \geq 0, \forall i \Rightarrow x_h \leq \frac{\bar{b}_i}{\bar{a}_{ih}}, \forall i : \bar{a}_{ih} > 0$$

$$t = \arg \min_{i=1 \dots m} \left\{ \frac{\bar{b}_i}{\bar{a}_{ih}} : \bar{a}_{ih} > 0 \right\}$$

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## Simplex method: check for unlimited LP

- Let  $x_h$ :  $\bar{c}_h < 0$ .

$$z = \bar{z}_B + \bar{c}_h x_h$$

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- If  $\bar{a}_{ih} \leq 0, \forall i = 1 \dots m$ , feasible solution with  $x_h \rightarrow +\infty$

### Condition of unlimited LP

There exists a basis such that

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write the LP in **canonical form** with respect to  $B$

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**if**  $(\bar{c}_j \geq 0, \forall j)$  **then**  $B$  is an optimal basis: **stop**

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Entering variable: any  $x_h : \bar{c}_h < 0$

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## Simplex tableau

- Represent the canonical form, can be used to operate Gauss-Jordan
- **Objective function as a constraint** (imposing the value of a new variable  $z$ ):

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightsquigarrow c_1x_1 + c_2x_2 + \dots + c_nx_n - z = 0$$

	$x_{B_1}$	$\dots$	$x_{B_m}$	$x_{N_1}$	$\dots$	$x_{N_{n-m}}$	$z$	$\bar{b}$
riga 0	$c_B^T$			$c_N^T$			-1	0
riga 1	$B$			$N$			0	$b$
$\vdots$							$\vdots$	
riga $m$							0	

- Elementary row ( $z$  included) operations: up to reading  $x_B$  (and  $z$ ) as functions of  $x_N$

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	$x_{B_1}$	...	$x_{B_m}$	$x_{N_1}$	...	$x_{N_{n-m}}$	$z$	$\bar{b}$
riga 0	0	...	0	<input type="checkbox"/>	...	<input type="checkbox"/>	-1	<input type="checkbox"/>
riga 1	1		0	<input type="checkbox"/>	...	<input type="checkbox"/>	0	<input type="checkbox"/>
⋮		⋮		<input type="checkbox"/>	...	<input type="checkbox"/>	⋮	<input type="checkbox"/>
riga $m$	0		1	<input type="checkbox"/>	...	<input type="checkbox"/>	0	<input type="checkbox"/>

### Tableau in canonical form

- Elementary row ( $z$  included) operations: up to reading  $x_B$  (and  $z$ ) as functions of  $x_N$

# Tableau and canonical form

	$x_{B_1}$	...	$x_{B_m}$	$x_{N_1}$	...	$x_{N_{n-m}}$	$z$	$\bar{b}$
$-z$	0	...	0	$\square$	...	$\square$	-1	$\square$
$x_{B_1}$	1		0	$\square$	...	$\square$	0	$\square$
$x_{B_i}$		$\ddots$		$\square$	...	$\square$	$\vdots$	$\square$
$x_{B_m}$	0		1	$\square$	...	$\square$	0	$\square$

$$z = \bar{z}_B + \bar{c}_{N_1} x_{N_1} + \bar{c}_{N_2} x_{N_2} + \dots + \bar{c}_{N_{(n-m)}} x_{N_{(n-m)}}$$

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# Tableau and canonical form

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$-z$	0	...	0	$\bar{c}_{N_1}$	...	$\bar{c}_{N_{n-m}}$	-1	$-\bar{z}_B$
$x_{B_1}$	1		0	$\bar{a}_{1N_1}$	...	$\bar{a}_{1N_{n-m}}$	0	$\bar{b}_1$
$x_{B_i}$		...		$\bar{a}_{iN_1}$	...	$\bar{a}_{iN_{n-m}}$	$\vdots$	$\bar{b}_i$
$x_{B_m}$	0		1	$\bar{a}_{mN_1}$	...	$\bar{a}_{mN_{n-m}}$	0	$\bar{b}_m$

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# Retrieving an initial feasible basis: **two-phases method**

- **Phase I:** solve an *artificial problem*

$$\begin{aligned} w^* = \min w = & \quad 1^T y = y_1 + y_2 + \cdots + y_m \\ \text{s.t.} \quad & \quad Ax + Iy = b \\ & \quad x, y \geq 0 \end{aligned} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}_+^m$$

If  $w^* > 0$ , the original problem is unfeasible, stop!

If  $w^* = 0$ , then  $y = 0$

- ▶ if some  $y$  in the (degenerate) basis, change basis to put all  $y$  out, thus obtaining an  $x_B$  feasible for the original problem!

- **Phase II:** solve the problem starting from the provided basis  $B$

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- ▶ if some  $y$  in the (degenerate) basis, change basis to put all  $y$  out, thus obtaining an  $x_B$  feasible for the original problem!

- **Phase II:** solve the problem starting from the provided basis  $B$

# Retrieving an initial feasible basis: **two-phases method**

- **Phase I:** solve an *artificial problem*

$$\begin{aligned} w^* = \min w = & \mathbf{1}^T y = y_1 + y_2 + \cdots + y_m \\ \text{s.t.} \quad & Ax + Iy = b \\ & x, y \geq 0 \end{aligned} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}_+^m$$

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# Simplex algorithm with matrix operations: revised simplex

$$\min z = c^T x$$

$$\text{s.t. } Ax = b$$

$$x \geq 0$$

standard form

$$\min z = c_B^T x_B + c_N^T x_N$$

$$\text{s.t. } Bx_B + Nx_N = b$$

$$x_B, x_N \geq 0$$

with (feasible) basis

$$x_B = B^{-1}b - B^{-1}Nx_N$$

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N)x_N$$

$$-z + \bar{c}_N^T x_N = -z_B$$

$$I x_B + \bar{N} x_N = \bar{b}$$

canonical (or tableau) form

$$\bullet \bar{b} = B^{-1}b$$

$$\bullet z_B = c_B^T \bar{b}$$

$$\bullet \bar{N} = B^{-1}N$$

$$\bullet \bar{c}_N^T = c_N^T - c_B^T B^{-1}N$$

# The (revised) simplex algorithm

- 1 Let  $\beta[1], \dots, \beta[m]$  be the column indexes of the **initial basis**
- 2 Let  $B = [A_{\beta[1]} | \dots | A_{\beta[m]}]$  and compute  $B^{-1}$  e  $u^T = c_B^T B^{-1}$
- 3 compute **reduced costs**:  $\bar{c}_h = c_h - u^T A_h$  for non-basic variables  $x_h$
- 4 If  $\bar{c}_h \geq 0$  for all non-basic variables  $x_h$ , **STOP**:  $B$  is **optimal**
- 5 Choose any  $x_h$  having  $\bar{c}_h < 0$
- 6 Compute  $\bar{b} = B^{-1}b = [\bar{b}_i]_{i=1}^m$  e  $\bar{A}_h = \bar{N}_h = B^{-1}A_h = [\bar{a}_{ih}]_{i=1}^m$
- 7 If  $\bar{a}_{ih} \leq 0, \forall i = 1 \dots m$ , **STOP**: **unlimited**
- 8 Determine  $t = \arg \min_{i=1 \dots m} \{\bar{b}_i / \bar{a}_{ih}, \bar{a}_{ih} > 0\}$
- 9 Change basis:  $\beta[t] \leftarrow h$ .
- 10 Iterate from Step 2

## Example

Solve:

$$\begin{array}{rllllll} \max & 3x_1 & + & x_2 & - & 3x_3 & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & - & x_3 & \leq & 2 & & \\ & x_1 & + & 2x_2 & - & 3x_3 & \leq & 5 & & \\ & 2x_1 & + & 2x_2 & - & x_3 & \leq & 6 & & \\ & x_1 \geq 0 & , & x_2 \geq 0 & , & x_3 \leq 0 & & & & \end{array}$$

Standard form

$$\begin{array}{rllllllllll} \min & -3x_1 & - & x_2 & - & 3\hat{x}_3 & & & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & \hat{x}_3 & + & x_4 & & & & = & 2 \\ & x_1 & + & 2x_2 & + & 3\hat{x}_3 & & & + & x_5 & & = & 5 \\ & 2x_1 & + & 2x_2 & + & \hat{x}_3 & & & & & + & x_5 & = & 6 \\ & x_1 & , & x_2 & , & \hat{x}_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$





## Iteration 1: steps 2–5

$$x_B^T = [x_4 \quad x_5 \quad x_6] \quad c_B^T = [0 \quad 0 \quad 0]$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [0 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 0]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - [0 \quad 0 \quad 0] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - 0 = -3$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - [0 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1 - 0 = -1 \quad h = 2 \text{ (} x_2 \text{ enters)}$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - [0 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = -3 - 0 = -3$$

## Iteration 1: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \quad \begin{array}{l} x_4 \\ x_5 \\ x_6 \end{array}$$

$$\bar{A}_h = B^{-1}A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$t = \arg \min \left\{ \frac{2}{1} \quad \frac{5}{2} \quad \frac{6}{2} \right\} = \arg \left( \frac{2}{1} \right) = 1 \quad \rightsquigarrow x_4 \text{ leaves}$$

$$\beta[1] = 2 \quad (\text{column 2 replaces } \beta[1] \text{ that was 4})$$

## Iteration 2: steps 2–5

$$x_B^T = [ x_2 \quad x_5 \quad x_6 ] \quad c_B^T = [ -1 \quad 0 \quad 0 ]$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [ -1 \quad 0 \quad 0 ] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = [ -1 \quad 0 \quad 0 ]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - [ -1 \quad 0 \quad 0 ] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - (-2) = -1$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - [ -1 \quad 0 \quad 0 ] \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = -3 + 1 = -2 \quad h = 3$$

( $\hat{x}_3$  enters)

$$\bar{c}_4 = c_4 - u^T A_4 = 0 - [ -1 \quad 0 \quad 0 ] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - (-1) = 1$$

## Iteration 2: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \begin{matrix} x_2 \\ x_5 \\ x_6 \end{matrix}$$

$$\bar{A}_h = B^{-1}A_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$t = \arg \min \left\{ \frac{2}{1} \quad \frac{1}{1} \quad X \right\} = \arg \left( \frac{1}{1} \right) = 2 \quad \rightsquigarrow x_5 \text{ leaves}$$

$$\beta[2] = 3 \quad (\text{column 3 replaces column } \beta[2] \text{ that was 5})$$

## Iteration 3: steps 2–5

$$x_B^T = [ x_2 \quad \hat{x}_3 \quad x_6 ] \quad c_B^T = [ -1 \quad -3 \quad 0 ]$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [ -1 \quad -3 \quad 0 ] \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} = [ 3 \quad -2 \quad 0 ]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - [ 3 \quad -2 \quad 0 ] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - (4) = -7 \quad h = 1$$

( $x_1$  enters)

It is not necessary to compute all reduced costs, stop as soon **one of them** is negative!

### Iteration 3: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \quad \begin{array}{l} x_2 \\ \hat{x}_3 \\ x_6 \end{array}$$

$$\bar{A}_h = B^{-1}A_1 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

$$t = \arg \min \left\{ \frac{1}{5} \quad X \quad X \right\} = \arg \left( \frac{1}{5} \right) = 1 \quad \rightsquigarrow x_2 \text{ leaves}$$

$$\beta[1] = 1 \quad (\text{column 1 replaces column } \beta[1] \text{ that was 2})$$

## Iteration 4

$$x_B^T = [ x_1 \quad \hat{x}_3 \quad x_6 ] \quad c_B^T = [ -3 \quad -3 \quad 0 ]$$

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [ -3 \quad -3 \quad 0 ] \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [ -6/5 \quad -3/5 \quad 0 ]$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - [ -6/5 \quad -3/5 \quad 0 ] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1 - (12/5) = 7/5$$

$$\bar{c}_4 = c_4 - u^T A_4 = 0 - [ -6/5 \quad -3/5 \quad 0 ] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - (6/5) = 6/5$$

$$\bar{c}_5 = c_5 - u^T A_5 = 0 - [ -6/5 \quad -3/5 \quad 0 ] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 - (3/5) = 3/5$$



## Optimal solution

Standard form (the one we solved by simplex method):

$$\bullet x_B^* \begin{bmatrix} x_1 \\ \hat{x}_3 \\ x_6 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix}$$

$$\bullet x_1^* = 1/5; x_2^* = 0; \hat{x}_3^* = 8/5; x_4^* = 0; x_5^* = 0; x_6^* = 4$$

$$\bullet z_{MIN}^* = c^T x^* = c_B^T x_B^T = \begin{bmatrix} -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix} = -27/5$$

Optimal solution for the initial problem:

$$\bullet x_1^* = 1/5$$

$$\bullet x_2^* = 0$$

$$\bullet x_3^* = -\hat{x}_3^* = -8/5$$

• first constraint satisfied with equality (since  $x_4^* = 0$ )

• second constraint satisfied with equality (since  $x_5^* = 0$ )

• third constraint satisfied with a slack of 4 (since  $x_6^* = 4$ )

$$\bullet z_{MAX}^* = -z_{MIN}^* = 27/5.$$

