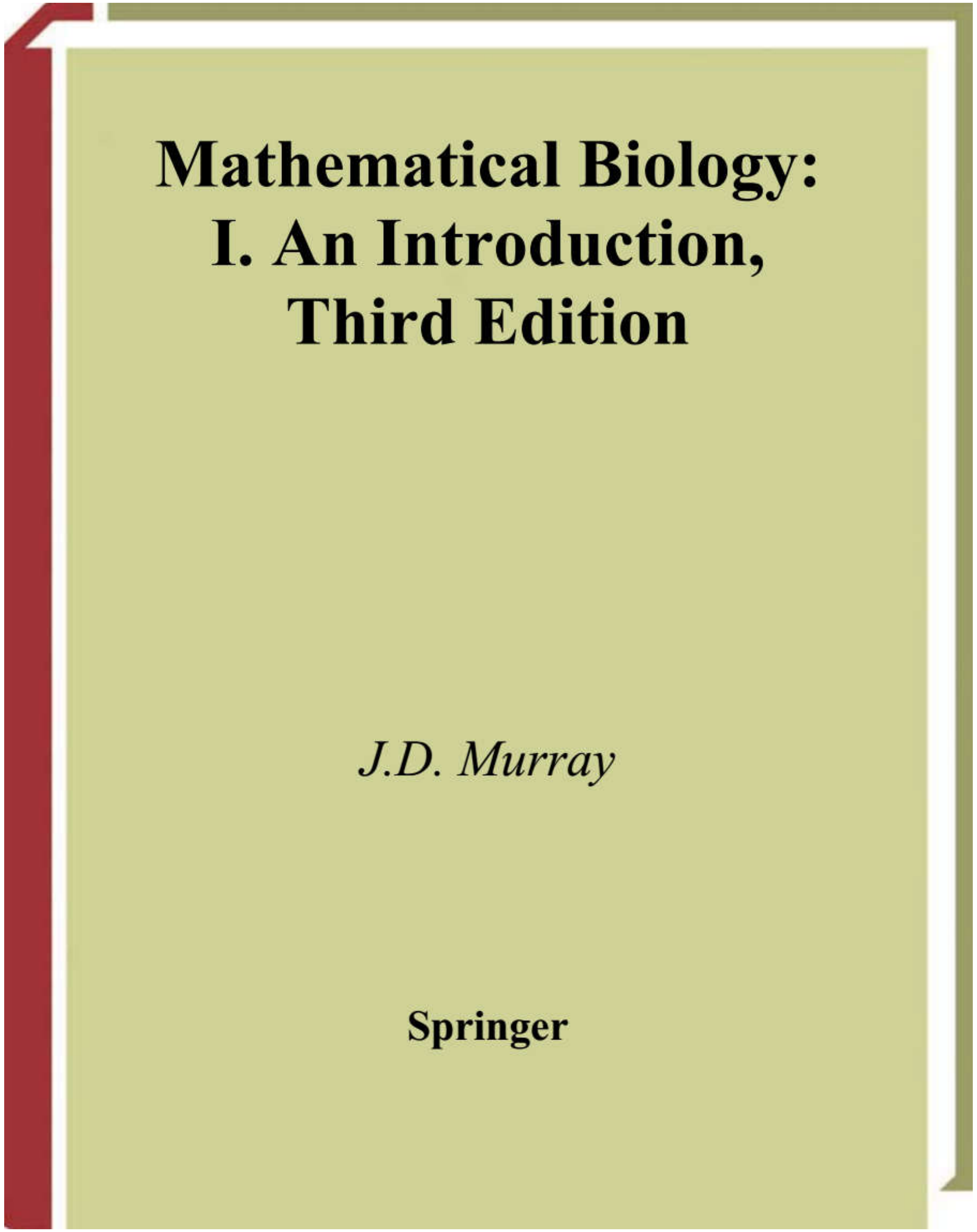


Dynamical systems and Phase Plane Analysis

We discuss here, only very briefly, general autonomous second-order ordinary differential equations of the form

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y). \tag{A.1}$$

Phase curves or phase trajectories of (A.1) are solutions of

$$\frac{dx}{dy} = \frac{f(x, y)}{g(x, y)}. \tag{A.2}$$

Through any point (x_0, y_0) there is a unique curve except at *singular points* (x_s, y_s) where

$$f(x_s, y_s) = g(x_s, y_s) = 0.$$

Thus, without loss of generality we now consider (A.2) to have a singular point at the origin; that is,

$$f(x, y) = g(x, y) = 0 \quad \Rightarrow \quad x = 0, y = 0. \tag{A.3}$$

If f and g are analytic near $(0, 0)$ we can expand f and g in a Taylor series and, retaining only the linear terms, we get

$$\frac{dx}{dy} = \frac{ax + by}{cx + dy}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(0,0)} \tag{A.4}$$

which defines the matrix A and the constants a, b, c and d. The linear form is equivalent to the system

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy. \tag{A.5}$$

Solutions of (A.5) give the parametric forms of the phase curves; t is the parametric parameter.

Let λ_1 and λ_2 be the eigenvalues of A defined in (A.4); that is,

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 = \frac{1}{2}(a + d \pm [(a+d)^2 - 4 \det A]^{1/2}). \qquad \text{(A.6)}$$

Solutions of (A.5) are then

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \mathbf{v}_1 \exp[\lambda_1 t] + c_2 \mathbf{v}_2 \exp[\lambda_2 t], \qquad \text{(A.7)}$$

where c_1 and c_2 are arbitrary constants and $\mathbf{v}_1$, $\mathbf{v}_2$ are the eigenvectors of A corresponding to λ_1 and λ_2 respectively; they are given by

$$\mathbf{v}_i = (1 + p_i^2)^{-1/2} \begin{pmatrix} 1 \\ p_i \end{pmatrix}, \quad p_i = \frac{\lambda_i - a}{b}, \quad b \neq 0, \quad i = 1, 2. \qquad \text{(A.8)}$$

Elimination of t in (A.7) gives the phase curves in the (x, y) plane.

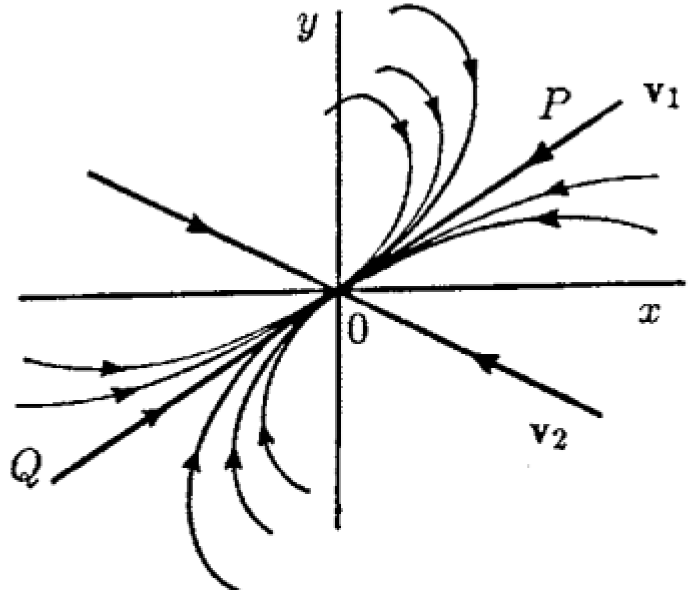
The form (A.7) is for distinct eigenvalues. If the eigenvalues are equal the solutions are proportional to $(c_1 + c_2 t) \exp[\lambda t]$.

Catalogue of (Linear) Singularities in the Phase Plane

(i) λ_1, λ_2 real and distinct:

$\lambda_2 < \lambda_1 < 0$ every solution tends to $(0, 0)$ as $t \to \infty$

This is called a *node* (Type I) singularity.

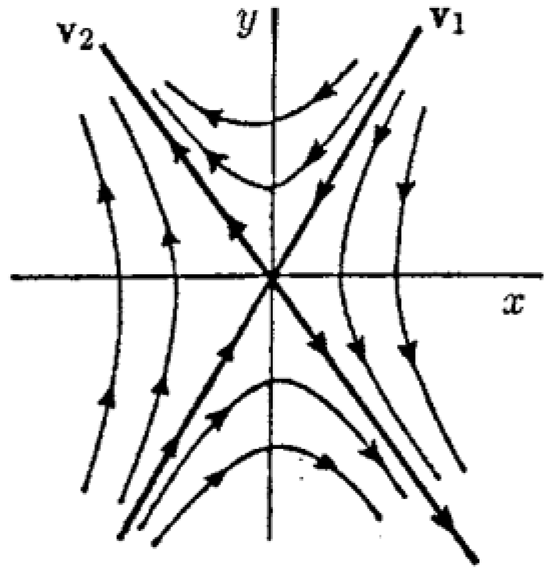


If $\lambda_1 > \lambda_2 > 0$ it is an unstable node; here $(x, y) \to (0, 0)$ as $t \to -\infty$.

(i) λ_1, λ_2 real and distinct:

(b) λ_1 and λ_2 have different signs. Suppose, for example, $\lambda_1 < 0 < \lambda_2$ then $\mathbf{v}_1 \exp[\lambda_1 t] v_1 \to 0$ along $\mathbf{v}_1$ as $t \to \infty$ while $\mathbf{v}_2 \exp[\lambda_2 t] \to 0$ along $\mathbf{v}_2$ as $t \to -\infty$.

This is a *saddle point* singularity.



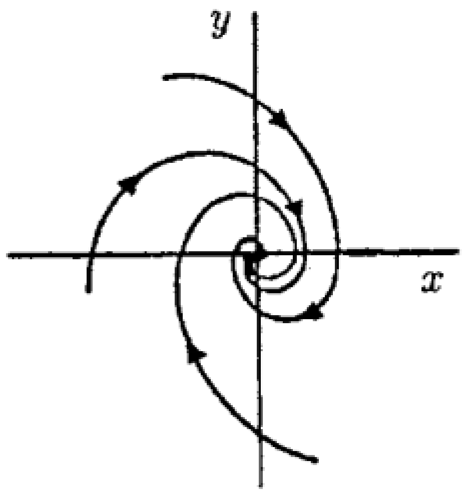
(b)

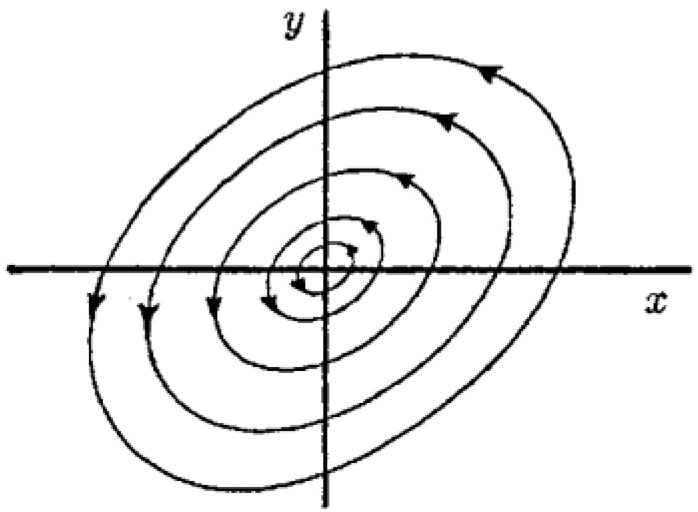
unstable: except strictly along $\mathbf{v}_1$ any small perturbation from (0, 0) grows exponentially.

(ii) λ_1, λ_2 complex: $\lambda_1, \lambda_2 = \alpha \pm i\beta, \beta \neq 0$. Solutions (A.7) here involve $\exp[\alpha t]$ $\exp[\pm i\beta t]$ which implies an oscillatory approach to $(0, 0)$.

(a) $\alpha \neq 0$. Here we have a *spiral*, which is stable if $\alpha < 0$ and unstable if $\alpha > 0$

(b) $\alpha = 0$. In this case the phase curves are ellipses. This singularity is called a *centre*

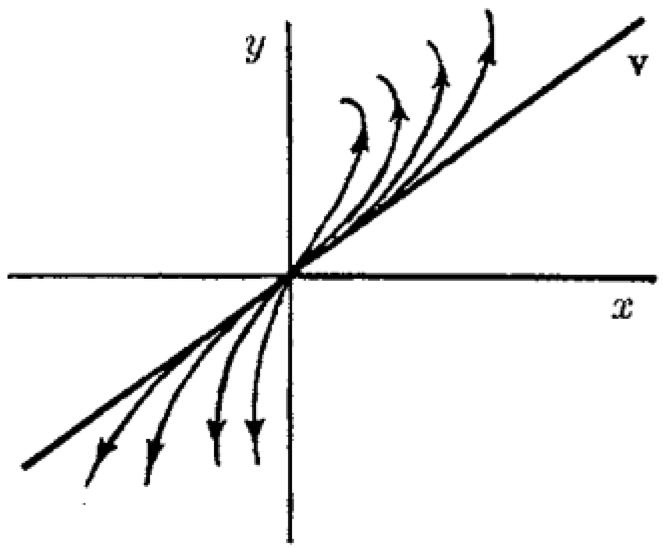


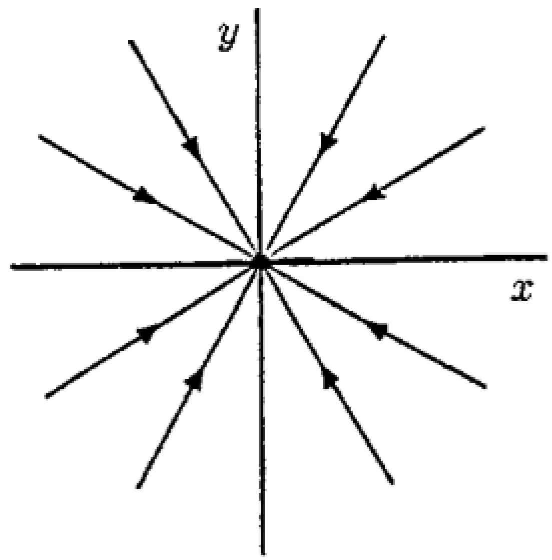


(iii) $\lambda_1 = \lambda_2 = \lambda$. Here the eigenvalues are *not* distinct.

(a) In general, solutions now involve terms like $t \exp[\lambda t]$ and there is only one eigenvector **v** along which the solutions tend to $(0, 0)$. The t in $t \exp[\lambda t]$ modifies the solution away from $(0, 0)$. It is called a *node* (Type II)

(b) If the solutions do not contain the $t \exp[\lambda t]$ term we have a *star* singularity, which may be stable or unstable, depending on the sign of λ.





Summary diagram showing how tr *A* and det *A*, where *A* is the linearisation matrix given by (A.4),

$$\frac{dx}{dy} = \frac{ax + by}{cx + dy}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(0,0)} \tag{A.4}$$

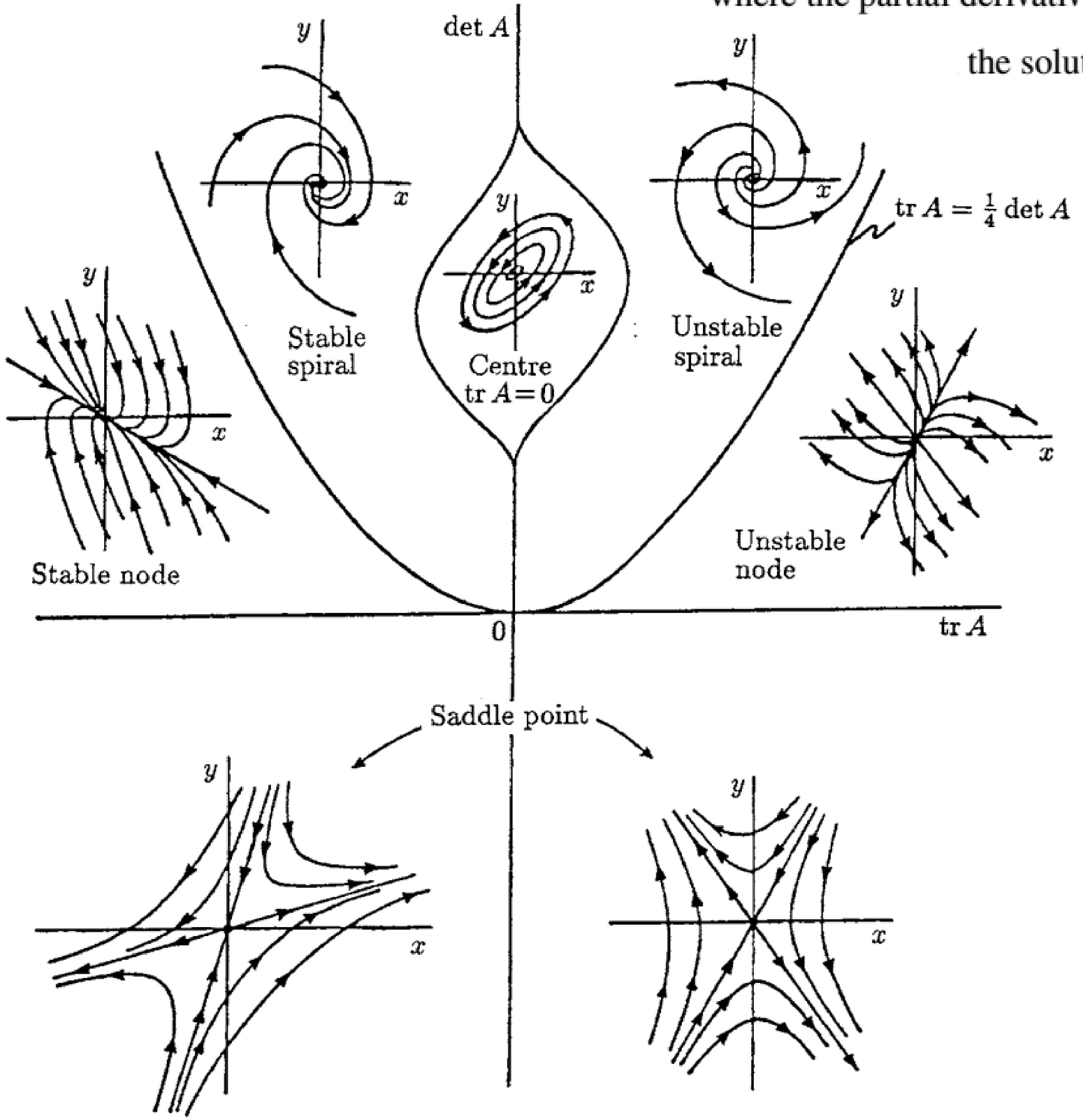
determine the type of phase plane singularity for (A.1).

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y). \tag{A.1}$$

Here $\det A = f_x g_y - f_y g_x$, $\operatorname{tr} A = f_x + g_y$

where the partial derivatives are evaluated at the singularities,

the solutions of $f(x, y) = g(x, y) = 0$.



Application:

an hydro-economic model for water sector investment, water security and risk

AGU PUBLICATIONS

Water Resources Research

RESEARCH ARTICLE

10.1002/2017WR020640

Special Section:

Socio-hydrology: Spatial and Temporal Dynamics of Coupled Human-Water Systems

Water security, risk, and economic growth: Insights from a dynamical systems model

Simon Dadson[1], Jim W. Hall[2], Dustin Garrick[2,3], Claudia Sadoff[4], David Grey[1], and Dale Whittington[5,6]

Investments in the physical infrastructure, needed for water resources management affect the economy in two distinct ways:

1. **by improving the factor productivity of water in multiple economic sectors, especially those that are water intensive such as agriculture and energy; and**

2. **by reducing acute and chronic harmful effects of water-related hazards like floods, droughts, and water-related diseases.**

The two influences are combined in a dynamical systems model of water-related investment, risk, and growth.

$$\frac{dK}{dt} = rK\left(1-\frac{K}{K_0}\right)\frac{N}{N_0} - sK - l_e\ K\left(1-\frac{N}{N_0}\right), \qquad (1)$$

$$\frac{dN}{dt} = sK - l_w N\left(1-\frac{N}{N_0}\right), \qquad (2)$$

Notation

$K(t)$	total country wealth at time t (\$).
$N(t)$	total investment in water-related assets at time t (\$).
t	time (year).
r	annual rate of return on investment.
s	fraction of national wealth invested in water-related assets annually.
l_e	fraction of national wealth exposed to water-related risks.
l_w	fraction of water-related assets exposed to water-related risks.
K_0	potential wealth when unrestricted by water-related factors (\$).
N_0	investment in water-related assets required to achieve K_0 (\$).

Nondimensionalization

$$\frac{d\alpha}{d\tau} = \alpha[(1-\alpha)\beta - \sigma - \lambda_e(1-\beta)], \qquad (3)$$

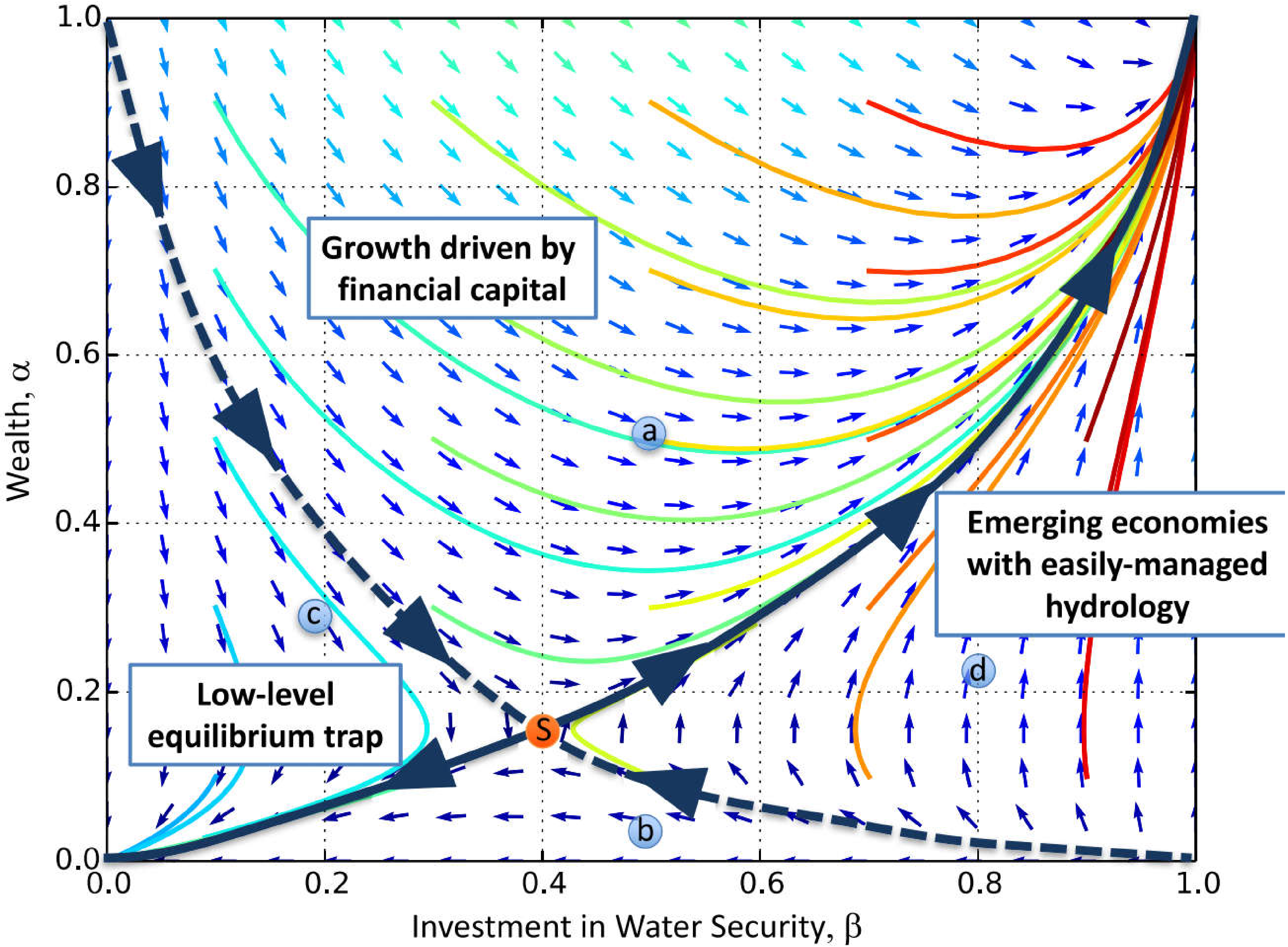
$$\frac{d\beta}{d\tau} = \frac{\sigma}{\phi}\alpha - \lambda_w \beta(1-\beta). \qquad (4)$$

$\tau = tr$, $\alpha = K/K_0$, $\beta = N/N_0$, $\sigma = s/r$, $\lambda_e = l_e/r$, $\lambda_w = l_w/r$, and $\phi = N_0/K_0$.

Investment in Water-Related Assets

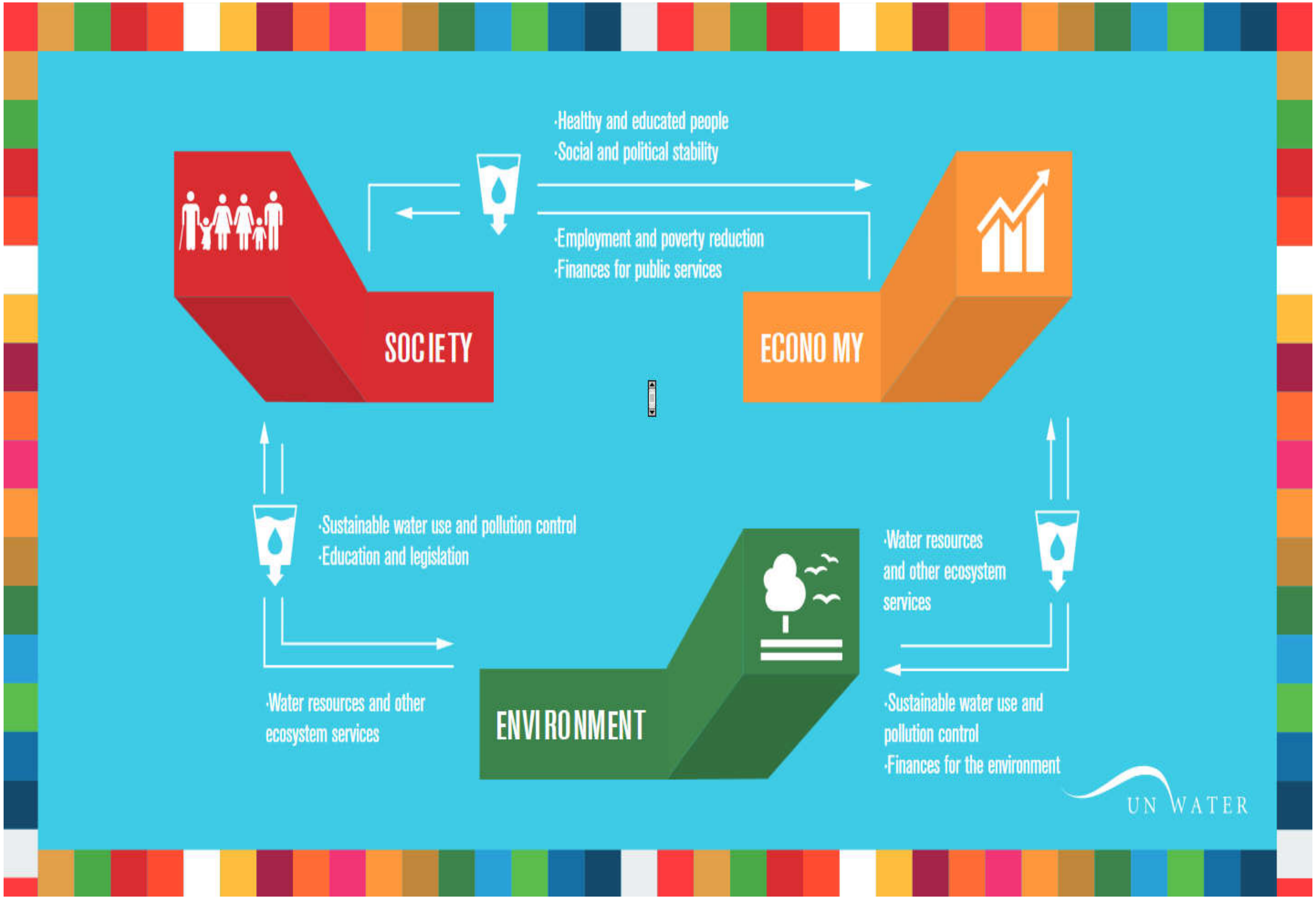
$$\sigma = 4\sigma_{max}\beta(1-\beta), \qquad (5)$$

The investment function σ is a function of the level of water security, β.

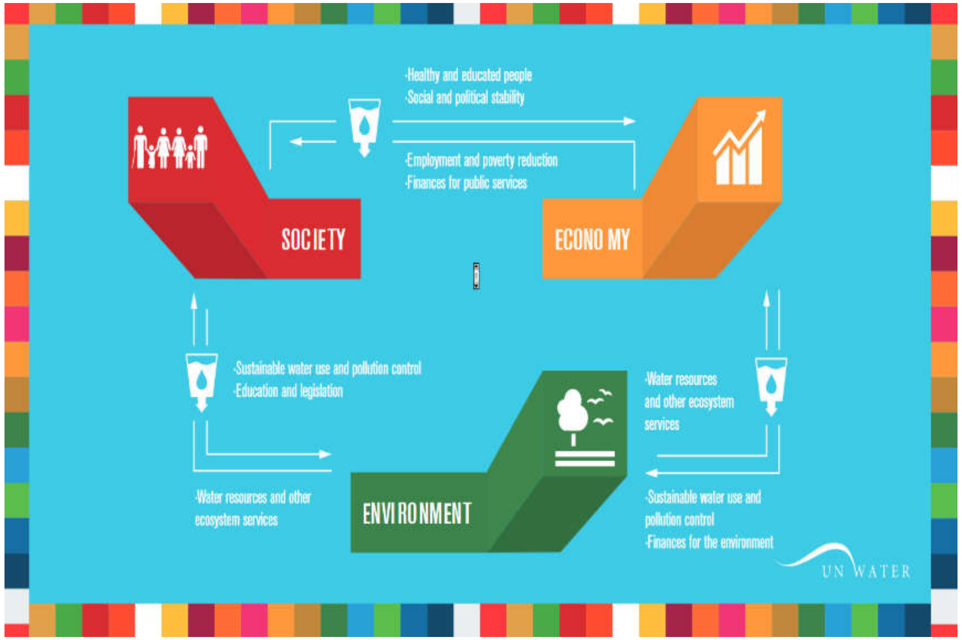


Growth driven by financial capital
Emerging economies with easily-managed hydrology
Low-level equilibrium trap
a
b
c
d
S
Wealth, α
Investment in Water Security, β
0.0
0.2
0.4
0.6
0.8
1.0

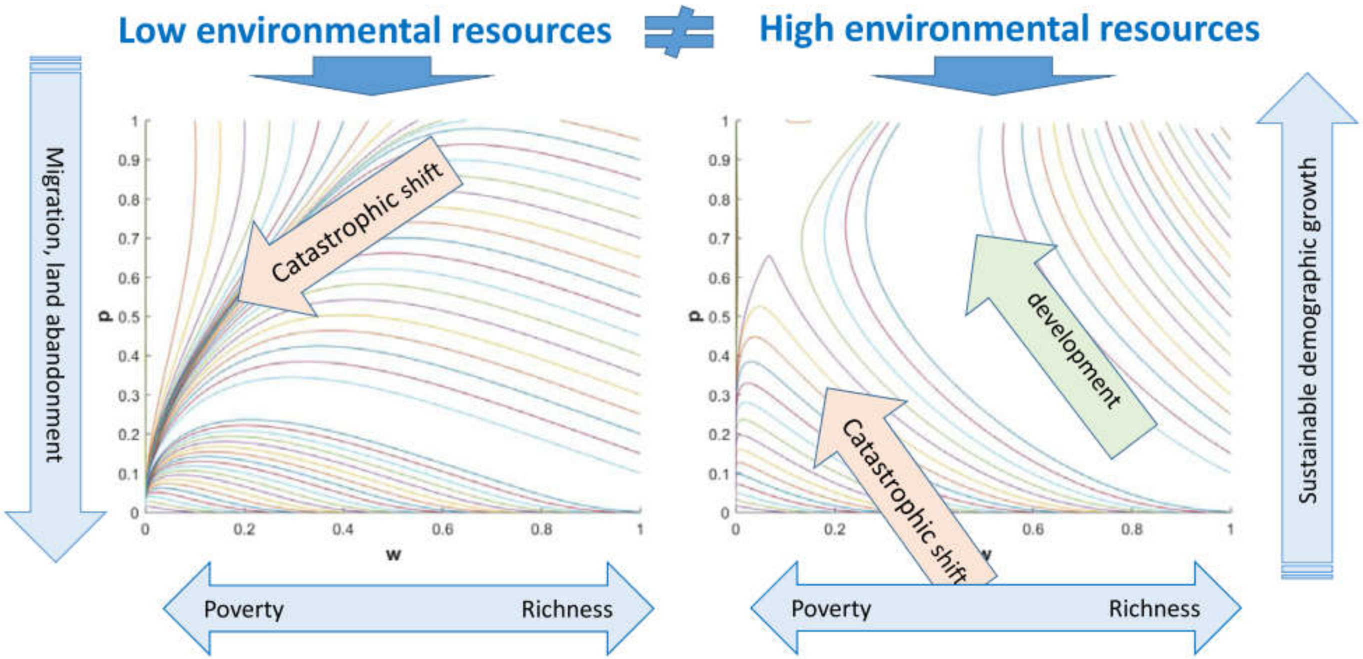
Water and the 3 Dimensions of Sustainable Development



UN-Water, 2016



Benessere, crescita demografica, tutela delle risorse ambientali



models of socio-ecological, catastrophic shifts and sustainable development

Science of the Total Environment 654 (2019) 890–894

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journal homepage: www.elsevier.com/locate/scitotenv

ELSEVIER

Dynamic models of socio-ecological systems predict catastrophic shifts following unsustainable development

Nadia Ursino

University of Padova, Dept. IMAGE, Italy

Interlaced dynamics of wealth *w*, environmental resources *e* and population *p*

Economic development

Investment in environmental resources conservation

Losses due to environmental risk

$$\frac{\partial w}{\partial t} = w\,(1 - w)p - \sigma w \;\exp\left(-\sigma\, e\right)\; p\,\frac{e}{e+k} - l\;\exp\left(-\lambda\,\frac{e}{p}\right)$$

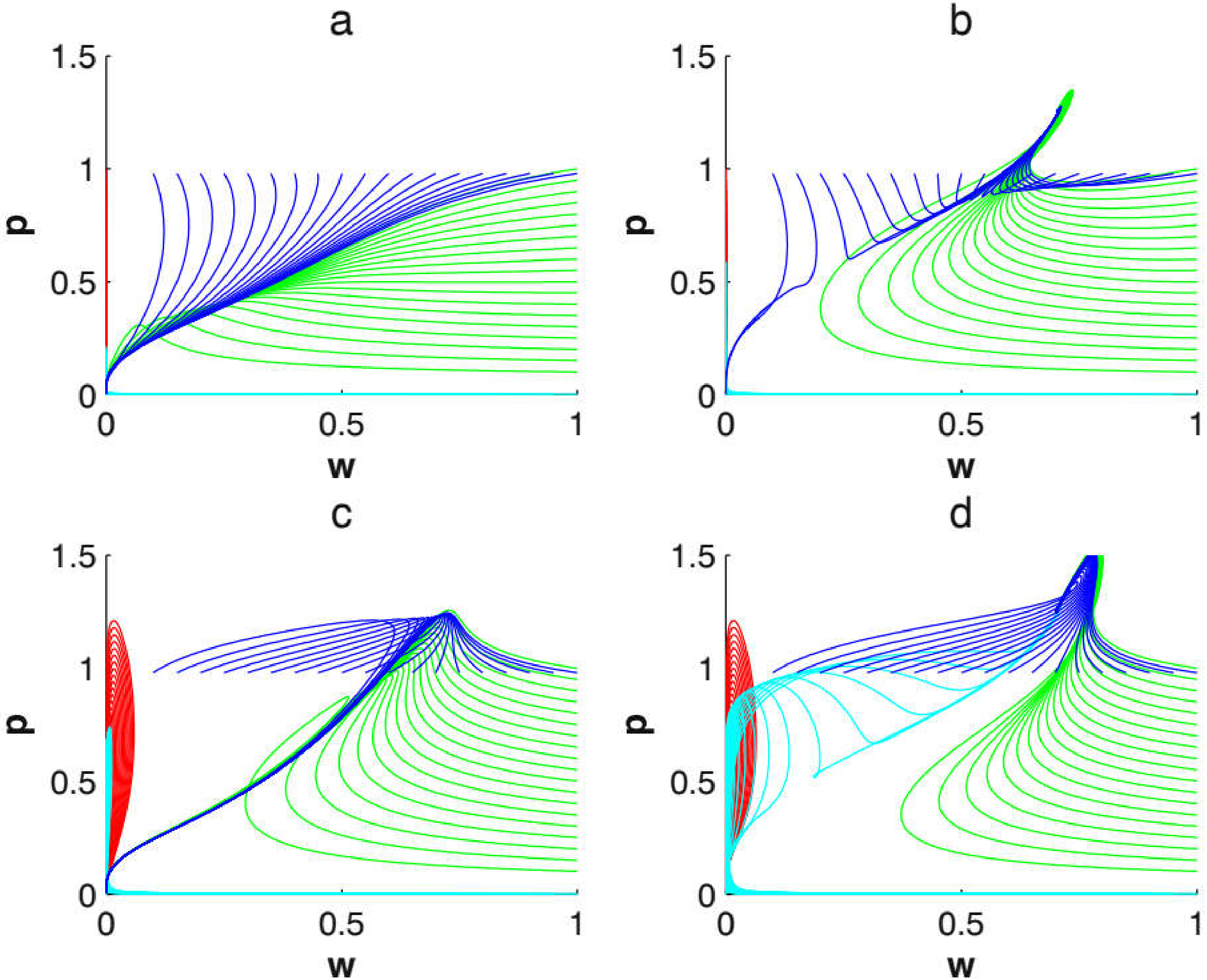
Effectivenss of investment

$$\frac{\partial e}{\partial t} = [\gamma\,\sigma w \;\exp\left(-\sigma\, e\right) - \delta]\; p\,\frac{e}{e+k}$$

Predator-prey dynamics

$$\frac{\partial p}{\partial t} = \delta\, p\,\frac{e}{e+k} - m\; p\;\exp\left(-\lambda\,\frac{e}{p}\right) - d\;p$$

migration due to environmental vulnerability (risk)

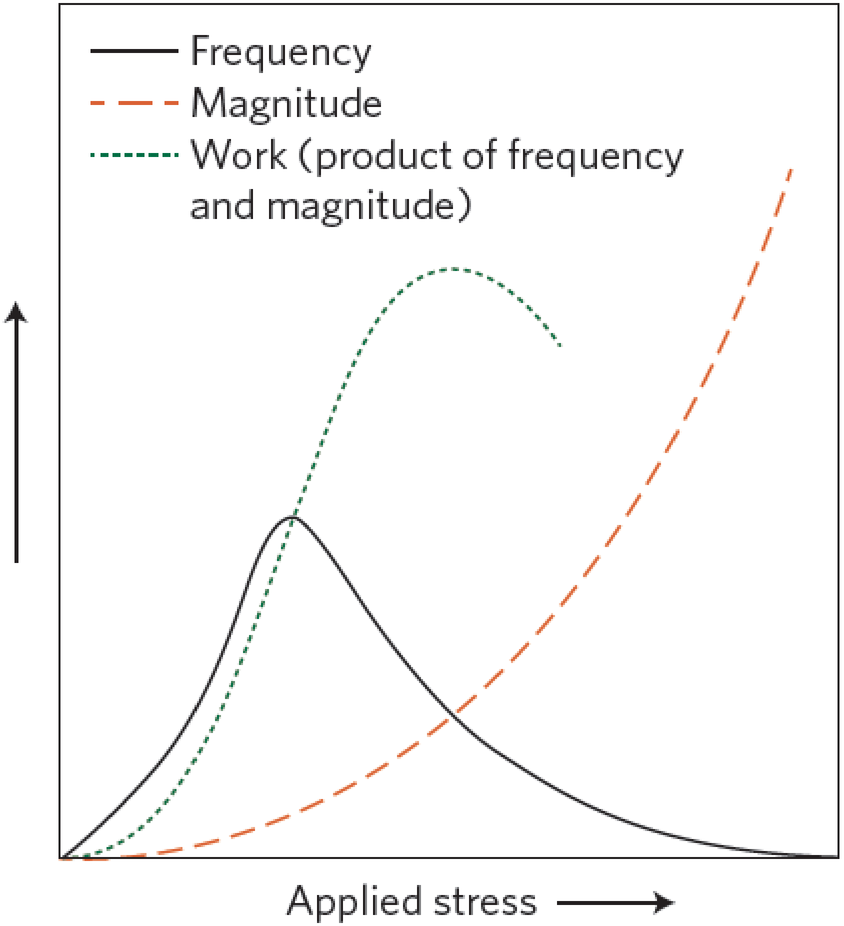


a) low natural resources and low effectiveness of investment
b) low natural resources and high effectiveness of investment
c) high natural resources and low effectiveness of investment
d) high natural resources and high effectiveness of investment

- The link between socioenvironmental dynamics and economic growth may be modelled.
- Dynamic models predict migration and land abandonment or sustainable development.
- Resources, politics and awareness affect socio-environmental dynamics.
- The system fate in the Anthropocene can be understood as prey-predator/breeder.
- The prey-predator/breeder model may aid decision making in favor of sustainability.

The catastrophic nature of humans

Richard Guthrie

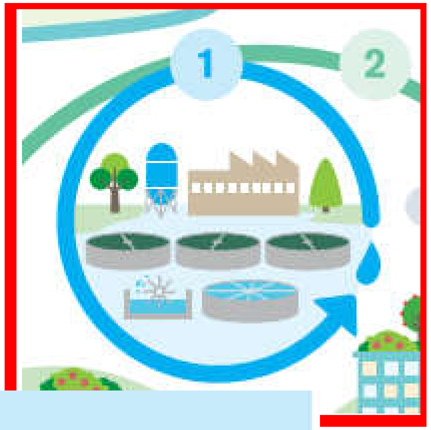


L’impatto che l’uomo ha sull’ambiente aumenta con il tempo e con la crescita demografica

The IWA Principles for Water-Wise Cities

La realizzazione di una gestione piu' sostenibile delle risorse idriche richiede:

- **Competenze**
- **Condivisione di esperienza**
- **Nuovi criteri progettuali**
- **interdisciplinarità**



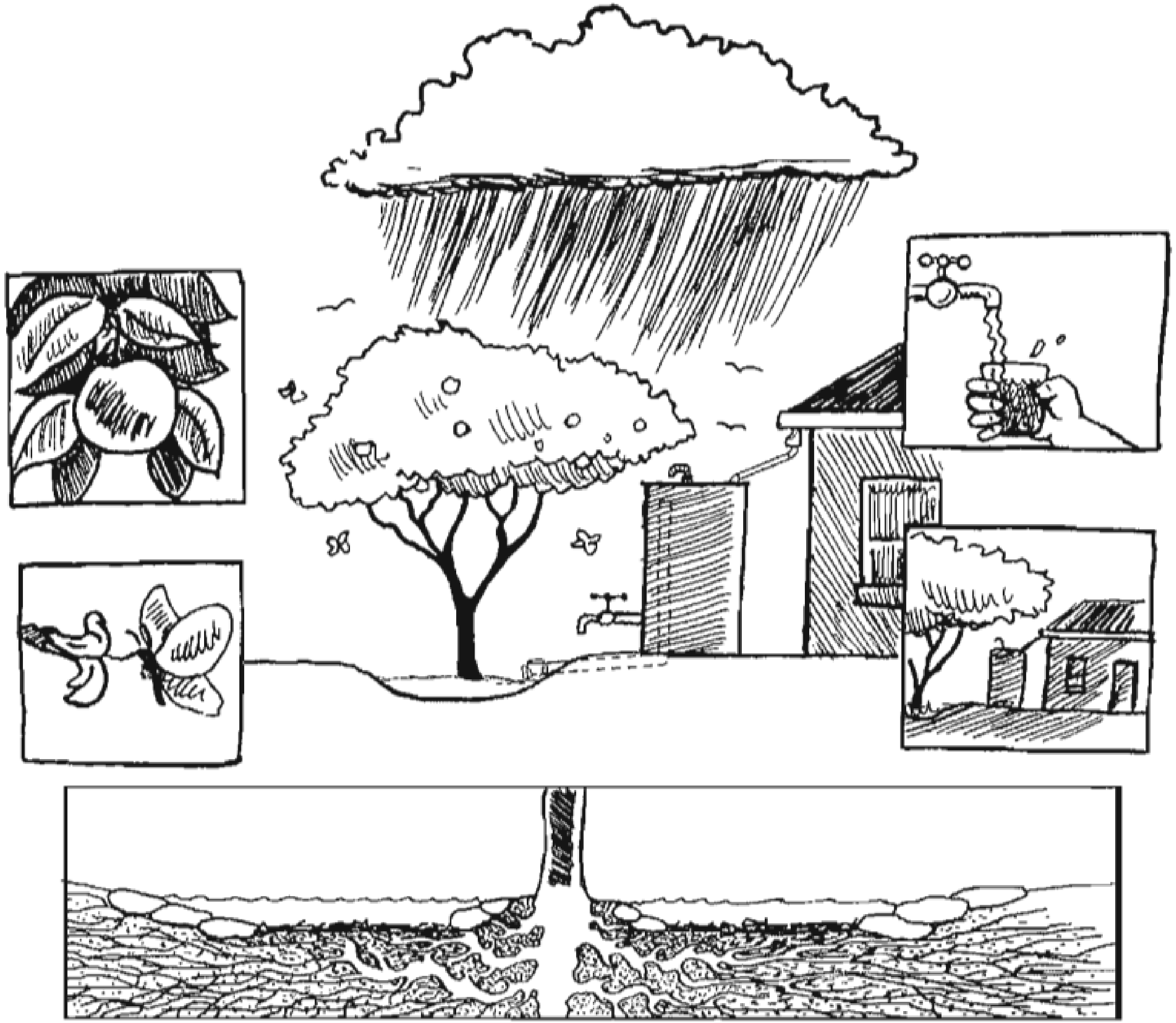
1 Regenerative Water Services

- **Replenish Waterbodies and their Ecosystems**
- **Reduce the Amount of Water and Energy Used**
- **Reuse, Recover, Recycle**
- **Use a Systemic Approach Integrated with Other Services**
- **Increase the Modularity of Systems and Ensure Multiple Options**

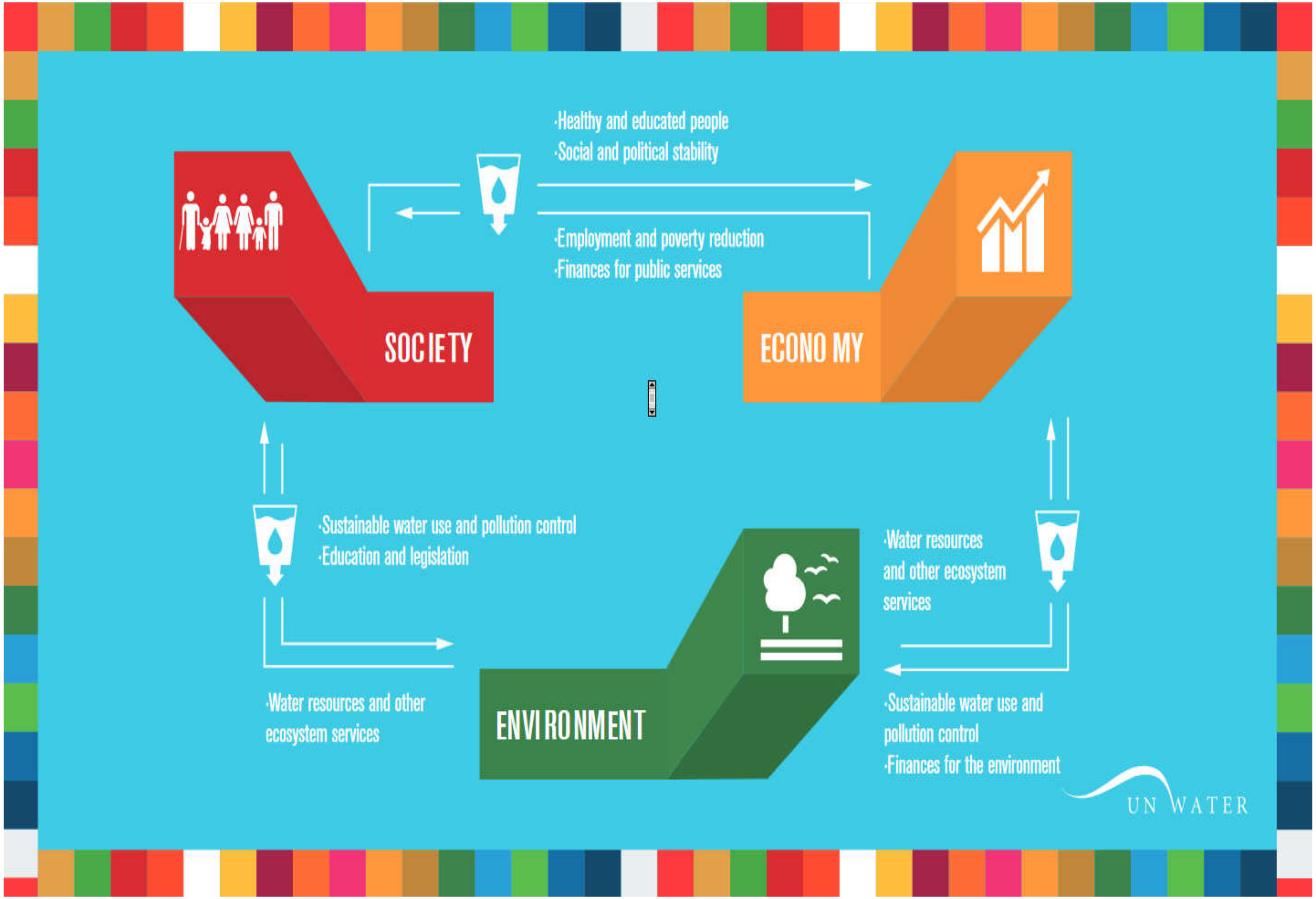
To fully realize the vision, increased capacities and competencies are needed, through sharing success stories from other cities, learning to work differently with new tools, pooling resources, and opening to other sectors' approaches and methods

(IWA)

7. DO MORE THAN JUST HARVEST WATER

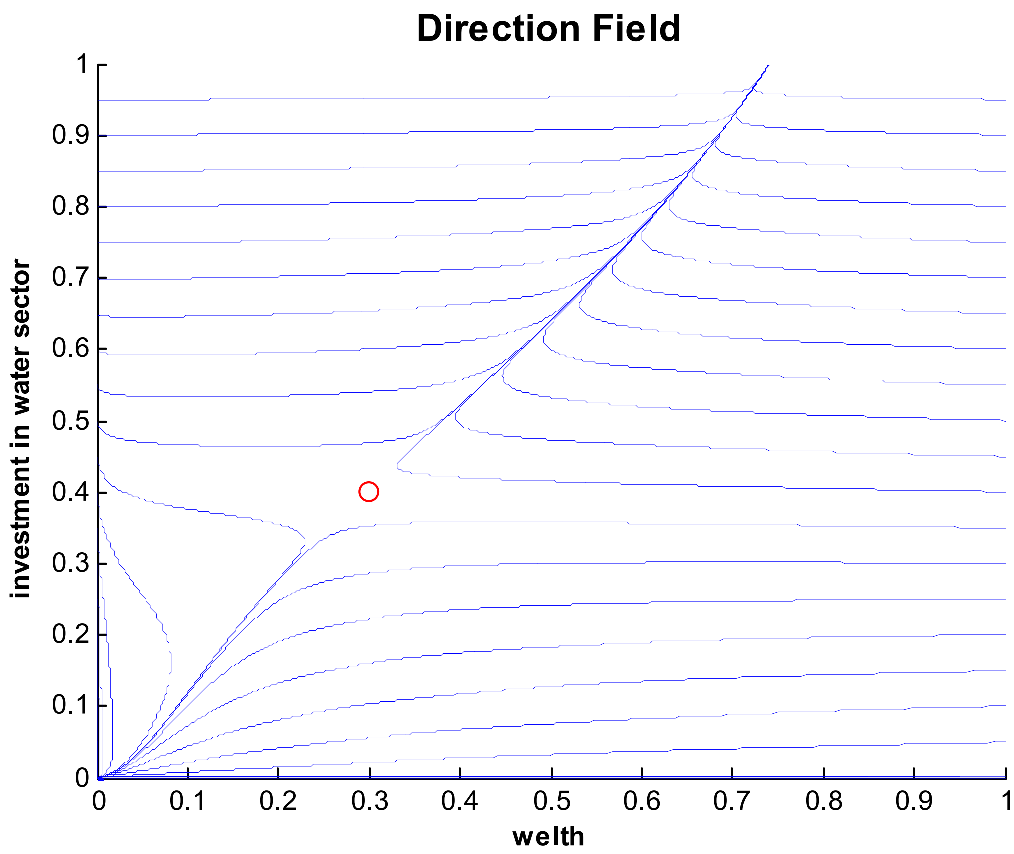


Water and the 3 Dimensions of Sustainable Development



UN-Water, 2016

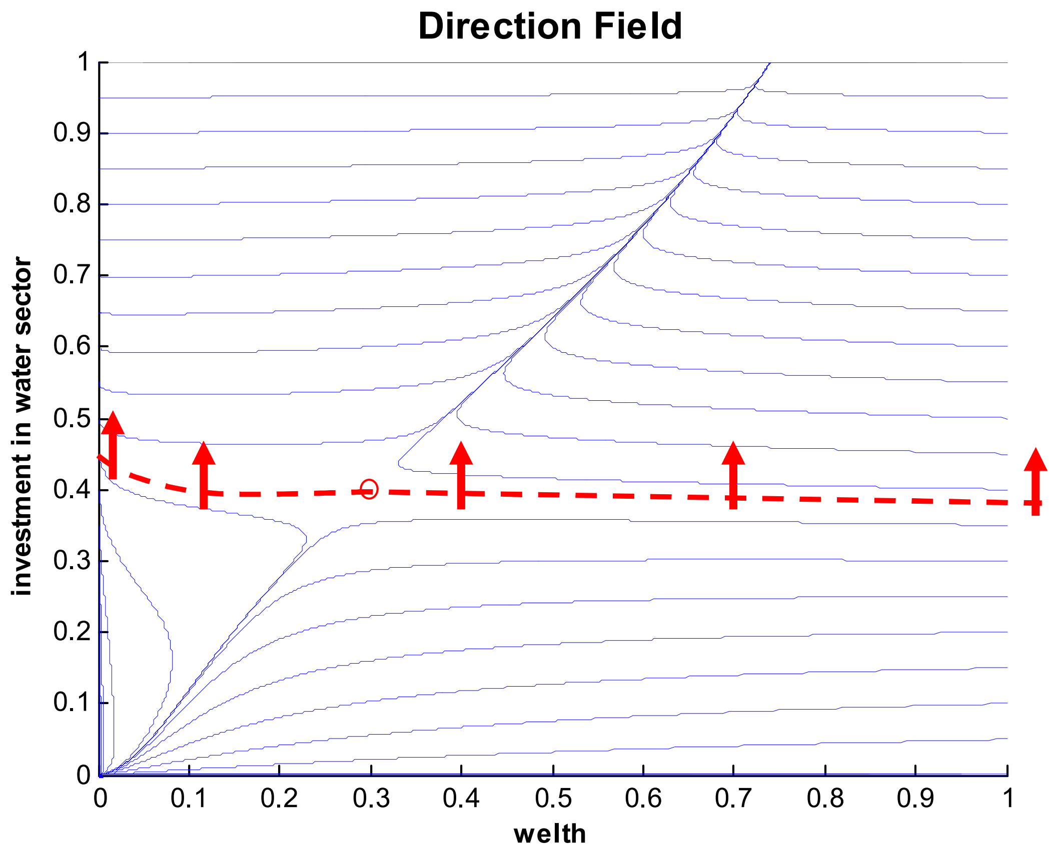
Investimento nel settore idrico:
realizzazione iniziative di salvaguardia ↑
Rischio (1-E)↓



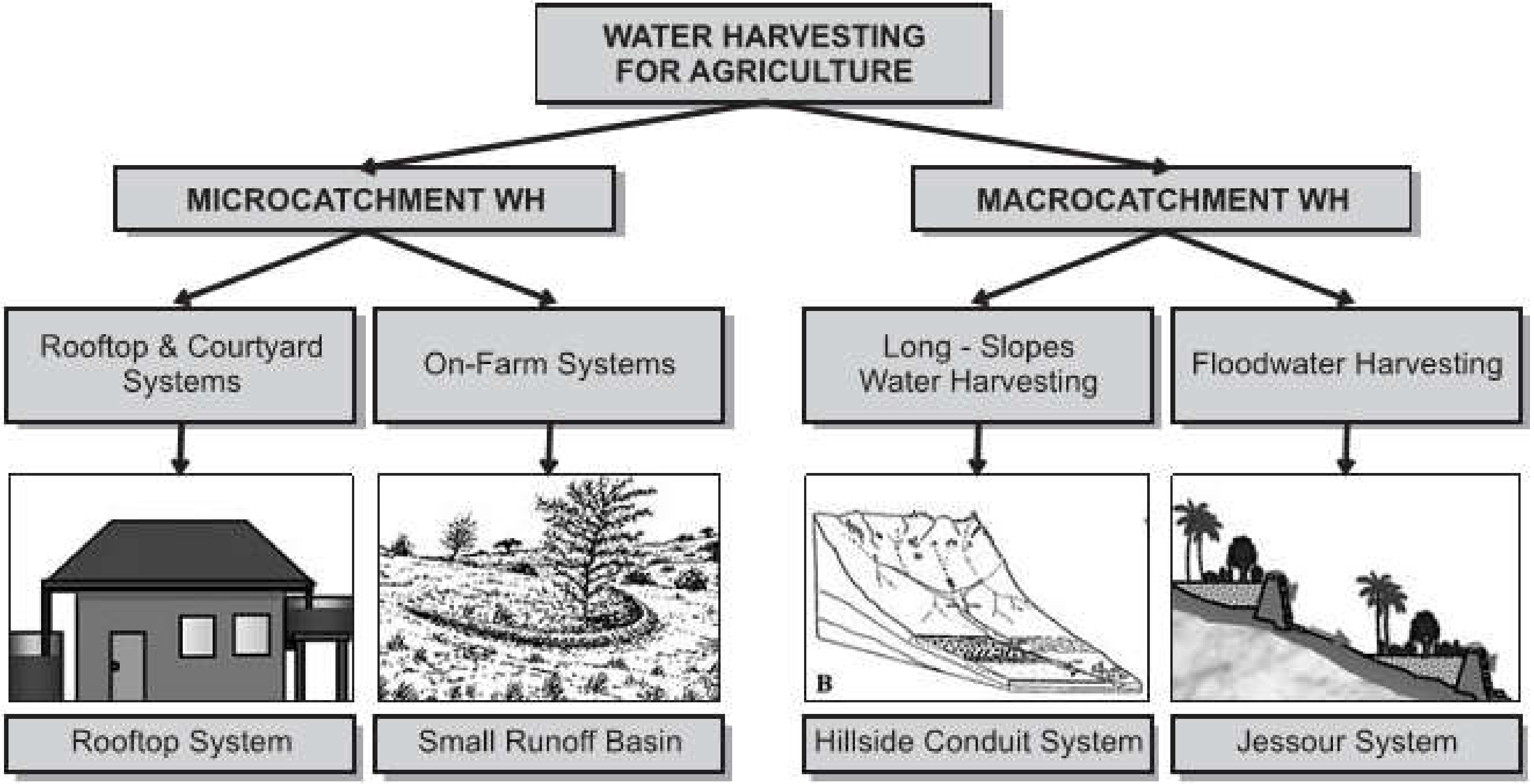
Ricchezza:
sviluppo↑
realizzazione iniziative di salvaguardia ↓
Rischio (1-E)↓

Investimenti nel settore idrico possono avere successo se…

di entità al di sopra di una certa soglia che dipende da fattori climatici, politici, sociali



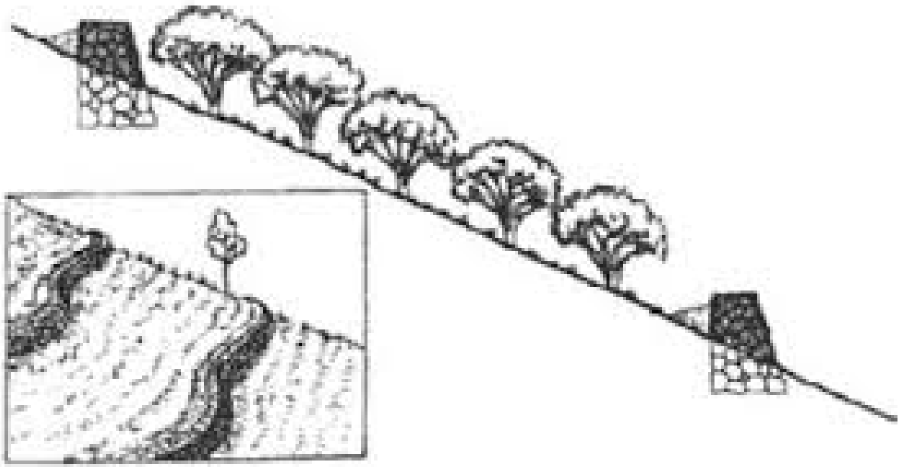
Overview and examples of water harvesting systems for agriculture

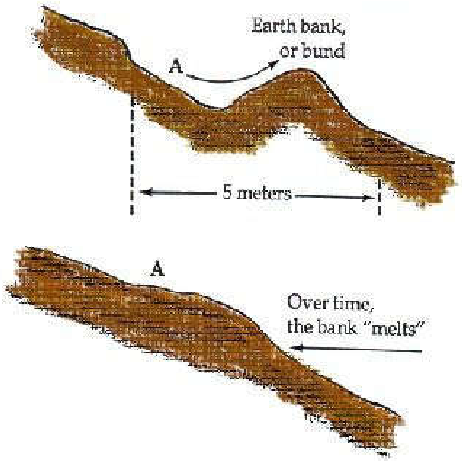


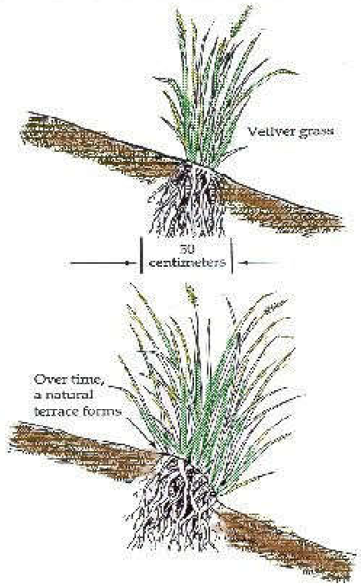
Theib Y. Oweis, Dieter Prinz, Ahmed Y. Hachum-Rainwater Harvesting for Agriculture in the Dry Areas-CRC Press (2012)

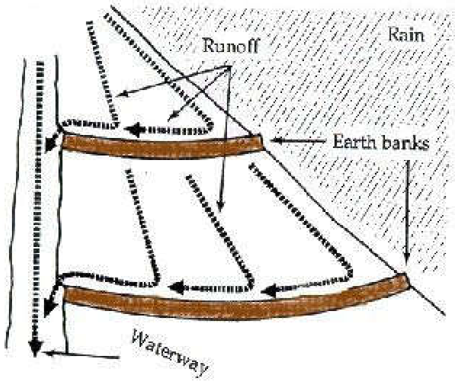
water-harvesting systems

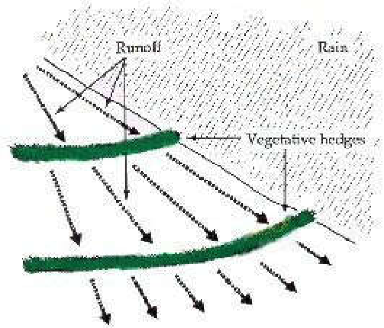
Contour stone walls



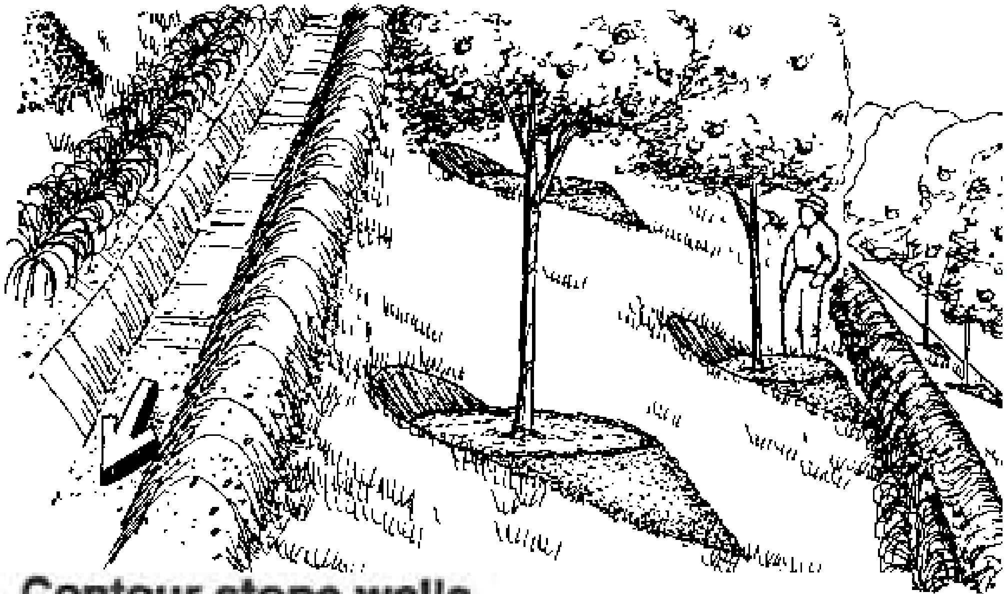




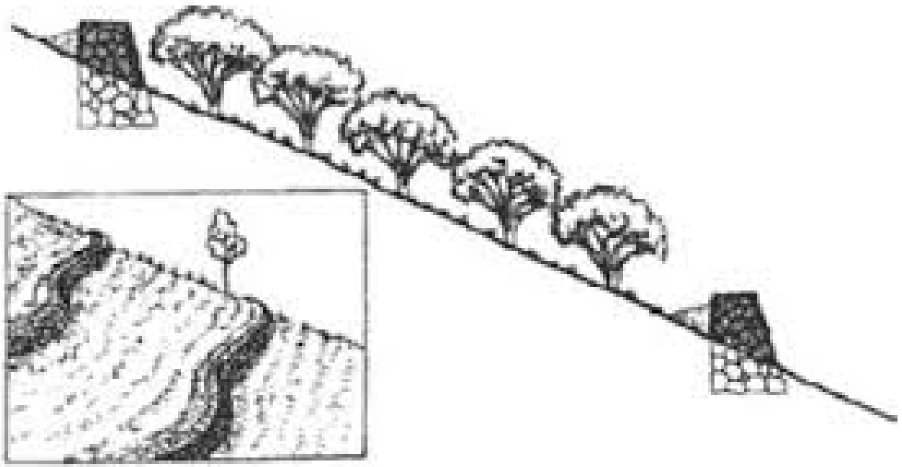


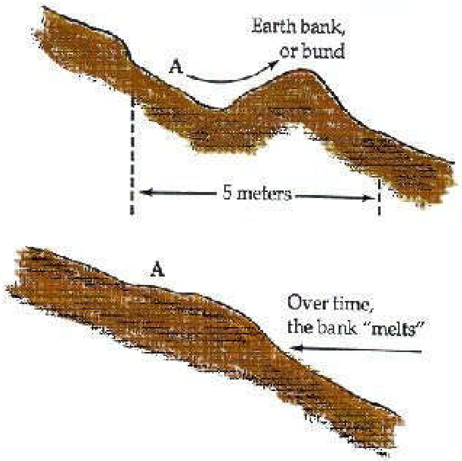


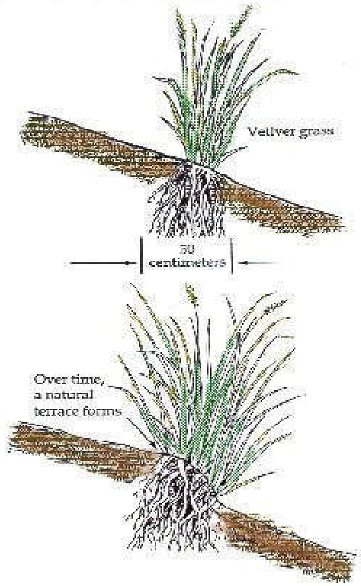
water-harvesting systems

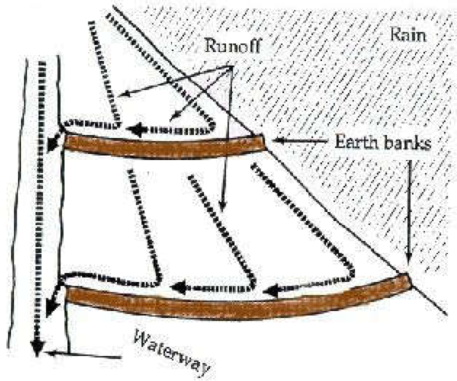


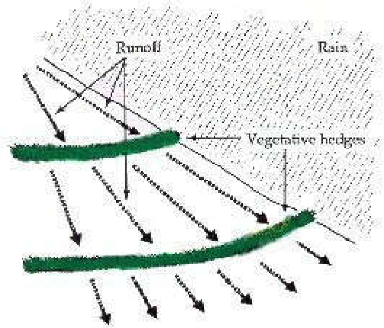
Contour stone walls

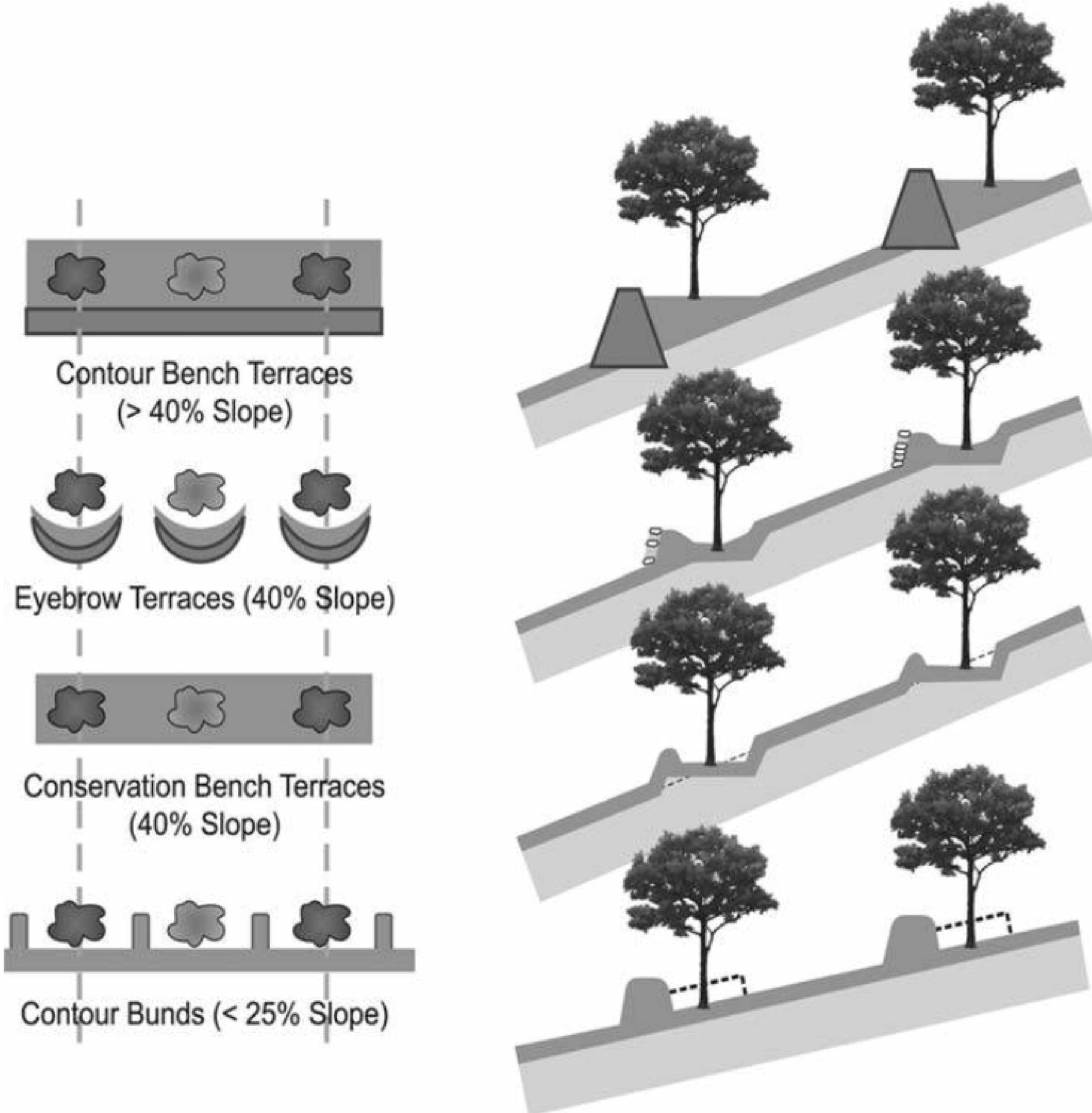




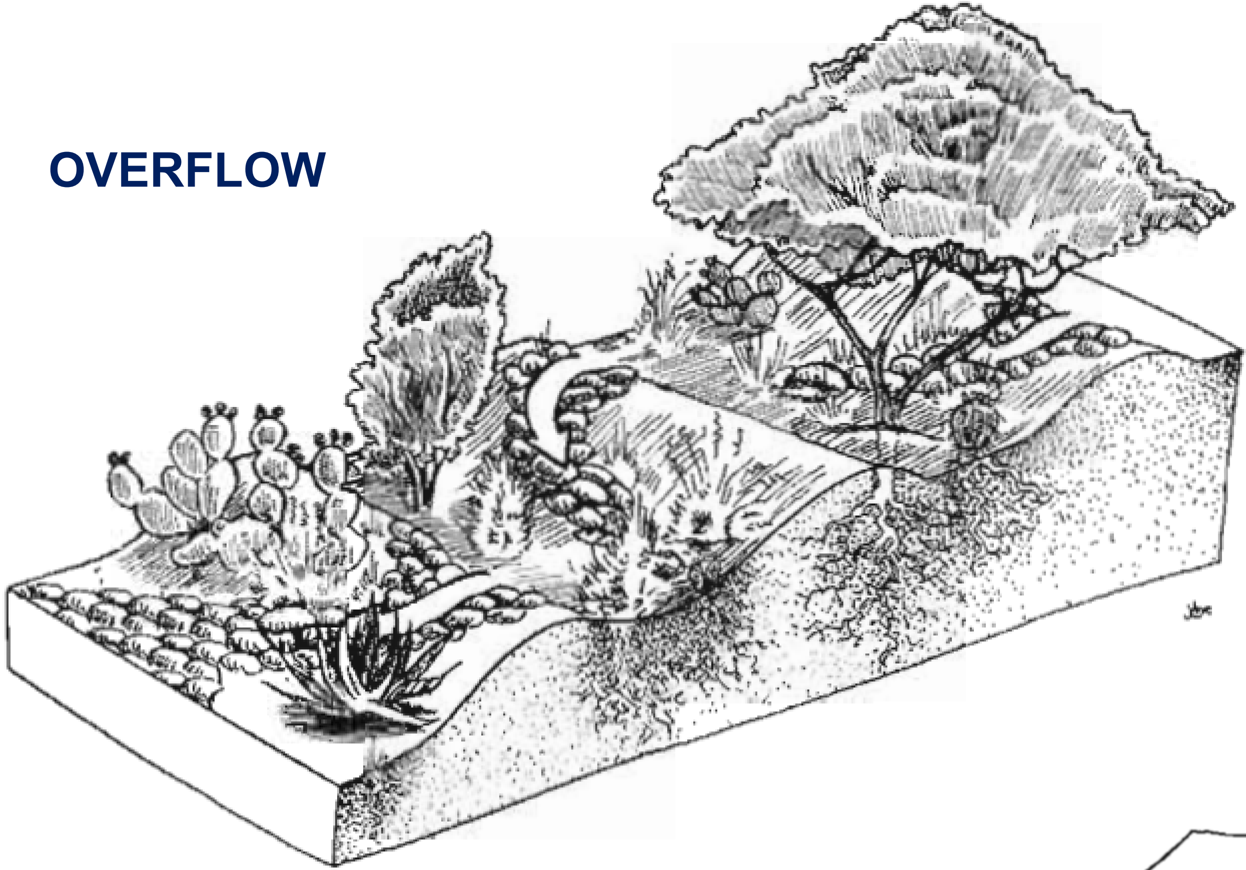


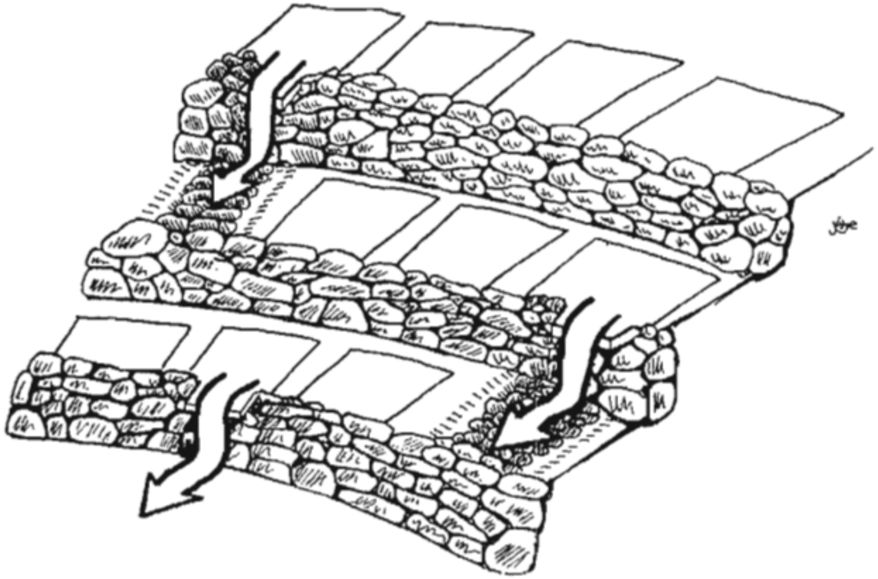





Contour Bench Terraces
(> 40% Slope)
Eyebrow Terraces (40% Slope)
Conservation Bench Terraces
(40% Slope)
Contour Bunds (< 25% Slope)

OVERFLOW



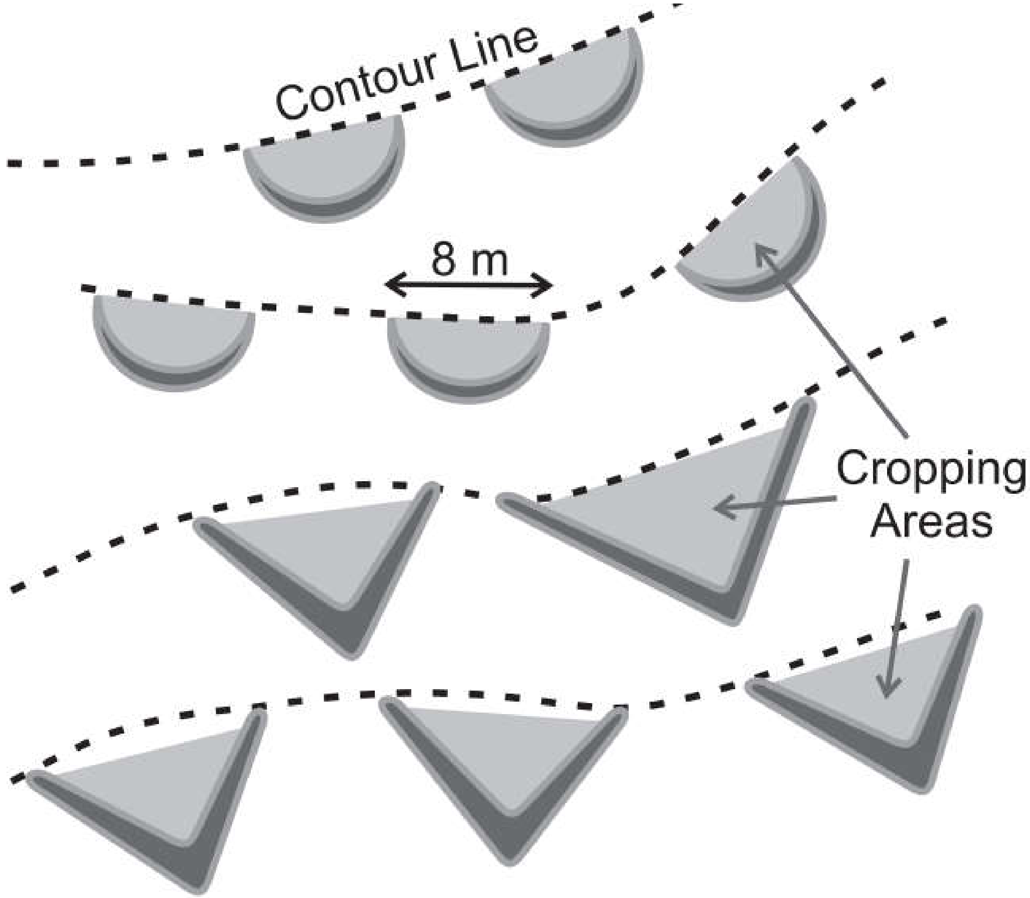


graded contour ridges with cross ties lower than the main ridges to retain water between the cross ties, but allow excess rainwater to flow between the ridges rather than spill over or break the main ridges

excessive rainwater breaching contour ridges at low points resulting in loss of rainwater by runoff and severe soil erosion - Mua, Malawi

Zaï pits or Tassa, for water harvesting - Illela, Niger

Examples of half moons for water harvesting - Illela, Niger


Contour Line
8 m
Cropping
Areas

Abraha Atsbeha-Etiopia

Abraha Atsbeha-Etiopia

Stagione piovosa 2-3 mesi
Trascorso: crescita demografica, abbandono dell'attività agricola a favore di pastorizia, desertificazione
1998: politiche di gestione sostenibile e recupero delle risorse ambientali, sistemi di infiltrazione, GW da -15 a -3 m spc
2012: UN Equator Prize
…trasferimento di conoscenza ad altri siti.

In the past, the highlands of northern Ethiopia have always belonged to the most drought-prone areas of the country. The rainy season lasts only two to three months. Strong population growth had caused massive deforestation and overgrazing, leading to land degradation and an increased vulnerability towards drought and famine. Like many other villages of Tigray, Abraha Atsbeha was chronically dependent on food aid.

In 1998, it was decided by the Ethiopian government that the people of the village were to be resettled to a different area. The land had become so barren that the government saw few other choices than to evacuate the valley. One alternative that was offered to the people: If they were to agree and strictly adhere to a new land management plan, carried out by their own workforce, the Ministry of Agriculture would support the restructuring with the help of international donors….

http://abahaui.tumblr.com/story





MEDICAM