SINGULARITIES OF SMOOTH MAPPINGS

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Abstract. In this seminar we want to provide a classification scheme for singularities of smooth mappings due to Thom and Boardman. According to the framework developed in [2] we remind the notions of Jet Bundles and Thom's transversality and then we introduce the intrinsic derivative of maps between bundles. We apply these tools to give a classification of singularities for a generic class of mappings and we illustrate applications to one and two dimensional cases. In particular, we show a result by Morse for the local behavior of a function near a non-degenerate critical point and the results by Whitney on the singularities of mappings between 2-manifolds (see [3]). In the end we define the Thom-Boardman stratification for the general case.
The restricted time frame prevents a presentation of a second scheme of classification, based on the "local ring" of a map, for which we refer to [1] and [2](Chapter VII).

Contents

1. Preliminaries

1.1. **Jet Bundles.**

Definition 1. Let X, Y be smooth manifolds and $p \in X$. Let $f, g : X \to Y$ be smooth mappings such that $f(p) = g(p) = q$.

(1) $f \sim_1 g$ at $p \overset{def}{\Leftrightarrow} df(p) = dg(p)$ as linear mappings $T_pX \to T_qY$.
(2) $f \sim_k g$ at $p \overset{def}{\Leftrightarrow} df(p) \sim_{k-1} dg(p)$ at every point of T_pX.
(3) Let $J^k(X,Y)_{p,q}$ be the set of the equivalence classes under the relation "$\sim_k$ at p" of mappings $f : X \to Y$, with $f(p) = q$.
(4) Set $J^k(X,Y) = \bigsqcup_{(p,q)\in X\times Y} J^k(X,Y)_{p,q}$. An element $\sigma \in J^k(X,Y)$ is called *k-jet.*
(5) Let σ a k-jet. Then there exist $p \in X$ and $q \in Y$ such that $\sigma \in J^k(X,Y)_{p,q}$: p is called *source of* σ and q is called *target of* σ. Moreover
$$\alpha :\ J^k(X,Y) \to X$$
$$\sigma \longmapsto (\text{source of } \sigma)$$
is the *source map* and
$$\beta :\ J^k(X,Y) \to Y$$
$$\sigma \longmapsto (\text{target of } \sigma)$$
is the *target map.*

Note that given $f : X \to Y$ is defined canonically the mapping
$$j^k f : X \to J^k(X,Y)$$
$$p \mapsto j^k f(p) = \text{the equivalence class of } f \text{ in } J^k(X,Y)_{p,f(p)}$$

Proposition 1. *Let $U \subset \mathbb{R}^n$ open, then $f, g : U \to \mathbb{R}^m$ are such that $f \sim_k g$ at $p \Leftrightarrow$ the Taylor expansions of f and g up to order k are identical at p. Hence we can write $j^k f(p) = (p, f(p), T_k f_1(p), ..., T_k f_m(p))$, where*
$$T_k f_i(p) = \sum_{|\alpha|>0} \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) \frac{(x-p)^\alpha}{\alpha!}.$$

Let A^k_n be the vector space of polynomials in n-variables of degree $\leq k$ which have their constant term equal to zero. One can choose as coordinates for A^k_n the coefficients of the polynomials. Then A^k_n is isomorphic to some Euclidean space $\mathbb{R}^l$. Let $B^k_{n,m} = \bigoplus_{i=1}^m A^k_n$.

Theorem 1. *Let X and Y be smooth manifolds of dimension n and m respectively. Then*

(1) *$J^k(X,Y)$ is a smooth manifold, with*
$$dim J^k(X,Y) = m + n + dim(B^k_{n,m}).$$
(2) *$\alpha : J^k(X,Y) \to X$, $\beta : J^k(X,Y) \to Y$ and $\alpha\times\beta : J^k(X,Y) \to X\times Y$ are submersions.*
(3) *If $g : X \to Y$ is smooth then $j^k g : X \to J^k(X,Y)$ is smooth.*

Proposition 2. *Let X, Y be smooth manifolds. Then $J^k(X,Y)$ is a fiber bundle over $X \times Y$, with fiber $B^k_{n,m}$.*

If $Y = \mathbb{R}^m$ then $J^k(X, \mathbb{R}^m)$ is a vector bundle over $X \times \mathbb{R}^m$, where the addiction of jets in $J^k(X, \mathbb{R}^m)_{p,q}$ is given by the addiction of functions representing these jets. Moreover $J^1(X, Y)$ is canonically isomorphic to $Hom(TX, TY)$, hence using this identification we can think of $J^1(X, Y)$ as a vector bundle over $X \times Y$ with fiber $Hom(\mathbb{R}^n, \mathbb{R}^m) \cong B^1_{n,m}$.

1.2. **The Whitney $\mathcal{C}^\infty$ Topology.** In this section our aim is to introduce a topology on $\mathcal{C}^\infty(X, Y)$.

Definition 2. Let X, Y be smooth manifolds.

(1) Let $k \in \mathbb{N}$. Let $U \subset J^k(X, Y)$ be an open set.

$$M(U) := \{f \in \mathcal{C}^\infty(X, Y) | j^k f(X) \subset U\}$$

(Note that $M(U \cap V) = M(U) \cap M(V)$)
The family

$$\left\{ M(U) \,\middle|\, U \underset{open}{\subset} J^k(X, Y) \right\}$$

form a basis for a topology on $\mathcal{C}^\infty$. This topology is called the Whitney $\mathcal{C}^k$ Topology.

(2) Let's denote with W_k the set of all open sets in $\mathcal{C}^\infty(X, Y)$ with Whitney $\mathcal{C}^k$ Topology, for every k. Then

$$W = \bigcup_{k=0}^{\infty} W_k$$

form a basis for Whitney $\mathcal{C}^\infty$ Topology on $\mathcal{C}^\infty(X, Y)$.

Theorem 2. *Let X, Y be smooth manifolds. Then $\mathcal{C}^\infty(X, Y)$ is a Baire space in the Whitney $\mathcal{C}^\infty$ Topology.*

(Note that by definition in a Baire space every residual subset, which is a countable intersection of open dense subsets, is dense).

2. Transversality

Definition 3. Let X and Y be smooth manifolds of dimensions n and m respectively. Let $f : X \to Y$ be a smooth mapping. Let W be a regular submanifold of Y and x a point in X. Then we say that f is transversal to W at x ($f \pitchfork W$ at x) if either

- $f(x) \notin W$ or
- $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W + Im\, df(x)$
  (where $df(x) : T_xX \to T_{f(x)}Y$)

We say that f is transversal to W if $f \pitchfork W$ at every point $x \in X$.

Proposition 3. *Let X, Y be smooth manifolds. Let $f : X \to Y$ be a differentiable mapping and let W be a regular submanifold of Y. Suppose that*

- $dimX + dimW < dimY$.
- $f \pitchfork W$.

The $f(X) \cap W = \emptyset$.

Lemma 1. *Let X, Y be smooth manifolds, $W \subset Y$ a submanifold, and $f : X \to Y$ smooth. Let $p \in X$ and $f(p) \in W$. Suppose there is a neighborhood U of $f(p)$ in Y and a submersion $\phi : U \to \mathbb{R}^k$ ($k = codim\, W$) such that $W \cap U = \phi^{-1}(0)$. Then $f \pitchfork W$ at p iff $\phi \circ f$ is a submersion at p.*

Remark 1. Note that such a neighborhood always exists. For there exists a chart (U, α), $\alpha : U \to \mathbb{R}^m$ and a decomposition of $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ so that $W \cap U = \alpha^{-1}(0_k \times \mathbb{R}^{m-k})$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^k$ be the projection on the first factor, then $\phi = \pi \circ \alpha$.

Proof. By assumption we have that $Ker\, d\phi(f(p)) = T_{f(p)}W$. So $f \pitchfork W$ at p if and only if

$$\begin{aligned} &T_{f(p)}Y = T_{f(p)}W + df(p)(T_pX) \\ \Leftrightarrow\ &T_{f(p)}Y = Ker\, d\phi(f(p)) + df(p)(T_pX) \end{aligned}$$

Since $d\phi(f(p))$ is surjective we have that $d(\phi \circ f)(p)$ is surjective $\Leftrightarrow$ the last equality holds. Thus $\phi \circ f$ is a submersion iff $f \pitchfork W$ at p. $\square$

Theorem 3. *Let $f : X \to Y$ be transversal to the regular submanifold W of Y (set $codim\, W = k$). Then $f^{-1}(W)$ is a regular submanifold of X of codimension k.*

Theorem 4 (Thom's Transversality Theorem)**.** *Let X and Y be smooth manifolds of dimensions n and m respectively. Let W be a submanifold of $J^k(X, Y)$. Let*

$$T_W = \left\{ f \in \mathcal{C}^\infty(X, Y) \,\middle|\, j^k f \pitchfork W \right\}.$$

Then T_W it's a residual subset of $\mathcal{C}^\infty(X, Y)$ in the Whitney $\mathcal{C}^\infty$ Topology.

3. Classification of Singularities

3.1. S_r **singularities.** We give a first classification of singularities of a smooth mapping. Let σ be in $J^1(X, Y)$. Then $\sigma \in J^1(X, Y)_{p,q}$ for some $(p, q) \in X \times Y$. We have that $J^1(X, Y) \cong Hom(TX, TY)$, then σ defines a unique linear map $T_pX \to T_qY$. If $f \in \mathcal{C}^\infty(X, Y)$ is a representative for σ (i.e. $j^1 f(p) = \sigma$) then this map is $df(p)$. Define:

- $rank\, \sigma = rank\, df(p)$
- $corank\, \sigma = l - rank\, df(p)$, with $l = min\{dimX, dimY\}$

Let:

$$S_r = \left\{ \sigma \in J^1(X, Y) \,\middle|\, corank\, \sigma = r \right\}$$

Let V, W be vector spaces. Let $S : V \to W$ be a linear map and define $corank\, S = q - rank\, S$, with $q = min\{dimV, dimW\}$. Define

$$L^r(V, W) = \left\{ S : V \to W \,\middle|\, corank\, S = r \right\}$$

Let $n = dimV$, $m = dimW$.

Proposition 4. *$L^r(V, W)$ is a submanifold of $Hom(V, W)$, with*

$$codimL^r(V, W) = (m - q + r)(n - q + r).$$

Theorem 5. *Let X, Y be manifolds, with $n = dimX$, $m = dimY$, $q = min\{dimX, dimY\}$. Then S_r is a submanifold of $J^1(X,Y)$, with $codimS_r = (n-q+r)(m-q+r)$. In fact, S_r is a subfiber-bundle of $J^1(X,Y)$ with fiber $L^r(\mathbb{R}^n, \mathbb{R}^m)$.*

Lemma 2. *$f : X \to Y$ is an immersion $\Leftrightarrow j^1f(X) \cap (\bigcup_{r\neq 0} S_r) = \emptyset$*

Let $Im(X,Y)$ be the subset of immersions in $\mathcal{C}^\infty(X,Y)$.

Theorem 6 (Whitney Immersion Theorem)**.** *Let X, Y be smooth manifolds with $dimY \geq 2{\cdot}dimX$. Then $Im(X,Y)$ is an open dense subset of $\mathcal{C}^\infty(X,Y)$ (in the Whitney $\mathcal{C}^\infty$ Topology).*

Definition 4. Let $f : X \to Y$. We say that f has a singularity of type S_r at $x \in X$ if $rank\, df(x) = q - r$, $(q = min\{dimX, dimY\})$.

Let $S_r(f) \subset X$ be the set of all singularities of f of type S_r.
Clearly $S_r(f) = (j^1f)^{-1}(S_r)$. Hence, by Thom Transversality Theorem, there exists an open and residual subset $G \subset \mathcal{C}^\infty(X,Y)$ such that $S_r(f)$ is a submanifold of X and $codimS_r(f) = codimS_r = (n-q+r)(m-q+r)$, $\forall f \in G$.

Definition 5. We will say that a mapping $f : X \to Y$ is *one generic* if $j^1f \pitchfork S_r$, for all r.

3.2. **The Intrinsic Derivative.** In this section we develop a technique due to Porteous for differentiating maps between vector bundles. This is the intrinsic derivative and it plays a rather important role in the Thom-Boardman theory. First we give a local definition of this notion and later a more global one.
Let X be a manifold and let $E = X \times \mathbb{R}^k$, $F = X \times \mathbb{R}^l$ be two product bundles over X. Let $\rho : E \to F$ be a vector bundle homomorphism. * Then for all $p \in X$, $\rho_p : \mathbb{R}^k \to \mathbb{R}^l$ is a linear map hence we can view ρ as a mapping

*Given two vector bundles $E \to X$, $F \to Y$, a map $\phi : E \to F$ is said to be a vector bundle homomorphism if

(1) $\exists$ a function $f : X \to Y$ called *base map* s.t.

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d] & F \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.
(2) ϕ is smooth.
(3) $\phi_p : E_p \to E_{f(p)}$ is a linear map.

If the two vector bundles have the same base manifold X, then the vector bundle homomorphisms are those with the identity as base map, i.e. those for which the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

$X \to Hom(\mathbb{R}^k, \mathbb{R}^l)$. Then for $p \in X$ makes sense to consider the differential

$$(d\rho)_p : T_pX \to T_pHom(\mathbb{R}^k, \mathbb{R}^l) = Hom(\mathbb{R}^k, \mathbb{R}^l)$$

Let $K_p = ker\rho(p)$ and $L_p = coker\rho(p)$. We define the intrinsic derivative of ρ at p, $(D\rho)_p$, in this trivial situation by the composite map

$$T_pX \to Hom(\mathbb{R}^k, \mathbb{R}^l) \to Hom(K_p, L_p)$$

where the second arrow is given by "'restricting and projecting"'. It's easy to see that the intrinsic derivative is invariant under vector bundle isomorphisms.

In general, let E and F be vector bundles over X and $\rho : E \to F$ be a vector bundle homomorphism. Then fixing $p \in X$, we can set $K_p = ker\rho(p)$ and $L_p = coker\rho(p)$. We may define the *intrinsic derivative of ρ at p*

$$(D\rho)_p : T_pX \to Hom(K_p, L_p)$$

by choosing a local trivialization of E and F in a neighborhood of p and computing as above. Since the definition is invariant under VB-isomorphism then the intrinsic derivative is independent of the choice of trivializations.

Now we give another definition of intrinsic derivative. Consider again the submanifold $L^r(V, W) \subset Hom(V, W)$ and let $A \in L^r(V, W)$, $K_A = kerA$, $L_A = cokerA$. Let N_A be the normal sapce to $L^r(V, W)$ at the point A, i.e.

$$N_A = \frac{T_A Hom(V, W)}{T_A L^r(V, W)}.$$

Then there is a canonical identification $N_A \cong Hom(K_A, L_A)$. This fact holds for vector bundles as well.

Let $E \to X$, $F \to X$ be vector bundles. Then we have the subfiber bundle $L^r(E, F)$ of $Hom(E, F)$, with generic fiber $L^r(E_x, F_x)$. Let $\sigma \in L^r(E_x, F_x)$. Since we are dealing with fiber bundles

$$N_\sigma = \frac{T_\sigma Hom(E, F)}{T_\sigma L^r(E, F)} = \frac{T_\sigma Hom(E_x, F_x)}{T_\sigma L^r(E_x, F_x)}$$

Then as in the case of vector spaces $N_\sigma \cong Hom(K_\sigma, L_\sigma)$, where $K_\sigma = ker\sigma$, $L_\sigma = coker\sigma$.

Let $\rho : E \to F$ be a vector bundles homomorphism. Looking at ρ as a map $X \to Hom(E, F)$ we set $\sigma = \rho_x$ and we assume $corank\sigma = r$ (hence $\sigma \in L^r(E, F)$). Then $(D\rho)_x$ is the composite map

$$T_xX \stackrel{(d\rho)_x}{\longrightarrow} T_\sigma Hom(E, F) \longrightarrow \frac{T_\sigma Hom(E, F)}{T_\sigma L^r(E, F)} \cong Hom(K_\sigma, L_\sigma)$$

One can immediately proves the following:

Proposition 5. *Suppose $\sigma = \rho_x$ is of corank r. Then the following are equivalent:*

1. *$(D\rho)_x : T_xX \to Hom(K_\sigma, L_\sigma)$ is surjective.*
2. *$\rho \pitchfork L^r(E, F)$ at x.*

Let $f : X \to Y$ be a smooth map. Consider the homomorphism between vector bundles

$$df : TX \to f^*TY$$

and fix $x \in X$.
We want to compute the intrinsic derivative

$$D(df)_x : T_xX \to Hom(K_x, L_x)$$

where $K_x = ker(df)_x$, $L_x = coker(df)_x$. We trivialize the tangent bundles by choosing coordinates $(x_1, ..., x_n)$ centered at x and $(y_1, ..., y_m)$ centered at $y = f(x)$. Moreover we choose this coordinates such that $(df)_x$ has the form

$$\left(\begin{array}{c|c} I_s & 0 \\ \hline 0 & 0 \end{array}\right)$$

In terms of these coordinates the last $m - s$ coordinate functions of f con be written in the form:

$$f_\alpha = \sum_{i,j=1}^{s} a_\alpha^{ij} x_i x_j + \sum_{j=1}^{s} \sum_{\beta=s+1}^{n} b_\alpha^{ij} x_\beta x_j + \sum_{\beta,\gamma=1}^{n} c_\alpha^{\beta\gamma} x_\beta x_\gamma + O(|x|^3)$$
$$\alpha = s+1, ..., m$$

We have that $Hom(T_xX \otimes K_x, L_x) \cong Hom(T_xX, Hom(K_x, L_x))$, hence the intrinsic derivative can be viewed as a map

$$D(df)_x : T_xX \otimes K_x \to L_x$$

Now we assume to use $x_{s+1}, .., x_n$ as linear coordinates in K_x and $y_{s+1}, .., y_m$ in L_x. Then $D(df)_x$ in coordinates is defined just by the last two terms of the expression for f_α.

$$\begin{aligned} D(df)_x :& T_xX \otimes K_x \to L_x \\ &(v, w) \mapsto \left(\sum_{l=1}^{n} \sum_{\beta=s+1}^{n} \frac{\partial^2 f_\alpha}{\partial x_\beta \partial x_l}(x) v_l w_\beta \right) \end{aligned}$$

with

$$\frac{\partial^2 f_\alpha}{\partial x_\beta \partial x_l}(x) = b_\alpha^{h\beta}, \qquad h = 1, .., s \ \beta = s+1, .., n \ \alpha = s+1, .., m$$
$$\frac{\partial^2 f_\alpha}{\partial x_\beta \partial x_l}(x) = c_\alpha^{h\beta} + c_\alpha^{\beta h}, \qquad h = 1, .., s \ \beta = s+1, .., n \ \alpha = s+1, .., m$$

Hence it's clear that the intrinsic derivative $D(df)_x$ is determined by the $2-$jet $j^2 f(x)$. Moreover if we restrict $D(df)_x$ to $K_x \otimes K_x$ we have a symmetric map, then it induces a mapping

$$\delta^2 f_x : K_x \circ K_x \to L_x$$

(where $K_x \circ K_x$ denotes the symmetric product).
Now let f be a real valued function and let x be a critical point of f. Then $K_x = T_xX$ and $L_x = \mathbb{R}$. The mapping

$$\delta^2 f_x : T_xX \circ T_xX \to \mathbb{R}$$

is called the Hessian of f at x.

3.3. **Morse Theory.** In this section we want to determine the structure of generic mappings of X into $\mathbb{R}$.
Since the dimension of $\mathbb{R}$ is 1, the only non-empty submanifolds of $J^1(X, \mathbb{R})$ of type S_r are S_0 and S_1. Thus p is a singularity (or a critical point) for $f : X \to \mathbb{R}$ if and only if $j^1 f(p) \in S_1$.

Definition 6. Let $f : X \to \mathbb{R}$ be a smooth mapping. We give the following definitions

- A point $p \in S_1(f)$ is said to be *nondegenerate* if $j^1 f \pitchfork S_1$ at p.
- f is a *Morse function* if all of its singularities are *nondegenerate*.

Just applying the Thom Transversality Theorem we have the following result.

Theorem 7. *Let X be a smooth manifold. Then the set of Morse functions is an open dense subset of $\mathcal{C}^\infty(X, \mathbb{R})$.*

Proposition 6. *Let $f : X \to \mathbb{R}$ be a smooth function. Then nondegenerate critical points of f are isolated.*

Proof. Let $p \in X$ be a nondegenerate critical point for f. Applying Lemma 1 we can say that there exist a neighborhood V of x s.t. $(j^1 f)^{-1}(S_1) \cap V$ is a submanifold of X of codimension equal to $codim S_1 = dim X$.
Then $(j^1 f)^{-1}(S_1) \cap V$ consists only of isolated points and hence there exist a neighbrohood $W \subset V$ of p s.t. $(j^1 f)^{-1}(S_1) \cap W = \{x\}$. $\square$

Proposition 7. *Let $f : X \to \mathbb{R}$ be smooth with a critical point at p. Then the symmetric bilinear form*

$$\delta^2 f_p : T_p X \circ T_p X \to \mathbb{R}$$

is nondegenerate iff p is a nondegenerate critical point of f.

Definition 7. Let p be a nondegenerate critical point of $f : X \to \mathbb{R}$. Then the *index* of f at p is the index of $\delta^2 f_p$.[†]

Theorem 8 (Morse)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$g(x_1, .., x_n) = c - (x_1^2 + \cdots x_k^2) + x_{k=1}^2 + \cdots x_n^2$$

where c is a constant. Then

(1) *g has a nondegenerate critical point of index k at the origin and has no other singularities.*

(2) *Let $f : X \to \mathbb{R}$ be a smooth with a nondegenerate critical point of index k at $p \in X$. Then there is a neighborhood U of p and a chart*

[†]

Proposition 8. *Let B a symmetric, nondegenerate, bilinear form on a real vector space V of dimension n. Then there is a basis $v_1, .., v_n$ of V and an integer $k \leq n$, called the* index *of B, such that $B(v_i, v_j) = s_i \delta_{ij}$, where $s_i = -1$ if $i \leq k$ and $s_i = 1$ if $i > k$.*

$\phi : U \to \mathbb{R}^n$ with $\phi(p) = 0$, such that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, where $c = f(p)$.

Note that in the coordinates $\phi_1, .., \phi_n$ defined in the neighborhood U of p, f has the normal form

$$f(x) = f(p) - (\phi_1^2(x) + \cdots + \phi_k^2(x)) + \phi_{k+1}^2(x) + \phi_n^2(x)$$

for all $x \in U$.

3.4. **The $S_{r,s}$ singularities.** Let $f : X \to Y$ be one-generic. We will denote with $S_{r,s}(f)$ the set of points where the map $f : S_r(f) \to Y$ drops rank s. We note that $codimS_r(f) > dimX - dimY$ hence $dimS_r(f) < dimY$. Thus we have that $x \in S_{r,s}(f)$ if and only if $x \in S_r(f)$ and $ker(df)_x$ intersects the tangent space to $S_r(f)$ in an s dimensional subspace. For example for maps between manifolds of dimension 2 we have to deal just with $S_{1,0}(f)$ and $S_{1,1}(f)$. Our goal for this section is to show that $S_{r,s}(f)$ are *generically* submanifolds and to compute their dimensions. The idea is to costruct global submanifold $S_{r,s}$ in $J^2(X,Y)$ such that

$$x \in S_{r,s}(f) \Leftrightarrow j^2f(x) \in S_{r,s}$$

First of all recall that $J^1(X,Y) \cong Hom(TX, TY)$ and hence $S_r \cong L^r(TX, TY)$. Given $\sigma \in S_r$ with source at x and target at y we can define the vector subspaces $K_\sigma = ker\sigma$ and $L_\sigma = coker\sigma$ in T_xX and T_yY respectively. Hence we have two vector bundles on S_r which we denote with K and L. As we have seen in section 3.2 the normal bundle to S_r in $J^1(X,Y)$ is canonically isomorphic to $Hom(K, L)$. Remember that we have a natural projection

$$\pi_{2,1} : J^2(X,Y) \to J^1(X,Y).$$

Then set $S_r^{(2)} = \pi_{2,1}^{-1}(S_r)$. Since the intrinsic derivative is determined by the 2-jet ,we have that the intrinsic derivative gives us a map of fiber bundles:

$$\begin{tikzcd} S_r^{(2)} \arrow[rr, "(\bullet)"] \arrow[dr, "\pi_{2,1}"'] & & Hom(K \circ K, L) \arrow[dl] \\ & S_r & \end{tikzcd}$$

Moreover it's clear from the calculations done in Section 3.2 that the top arrow is surjective. Now we recall some standard algebraic facts, let V and W be vector spaces, then:

(1) $V \otimes V = (V \circ V) \oplus (V \wedge V)$. So there is a canonical projection $\pi : V \otimes V \to V \circ V$ whose kernel is $V \wedge V$.

(2) $Hom(V \otimes V, W) \cong Hom(V, Hom(V, W))$.

Consider the map

$$Hom(V \circ V, W) \to Hom(V \otimes V, W) \to Hom(V, Hom(V, W))$$

where the first arrow is given by $A \mapsto A{\cdot}\pi$ and the second arrow by $B \mapsto \phi_{B}$, $\phi_{B}(u)(v) = B(u \otimes v)$.
Let $Hom(V \circ V, W)_s$ be the pre-image of $L^s(V, Hom(V, W))$ under this map, then the following proposition holds:

Proposition 9. *$Hom(V \circ V, W)_s$ is a submanifold of $Hom(V \circ V, W)$ of codimension*

$$\frac{l}{2}k(k+1) - \frac{l}{2}(k-s)(k-s+1) - s(k-s) \quad (\star)$$

where $k = dimV$, $l = dimW$.

Given two vector bundles $E \to X$, $F \to X$ let $Hom(E \circ E, F)_s$ be the fiber bundle whose fiber is $Hom(E_p \circ E_p, F_p)$. By the previous proposition then we have that $Hom(E \circ E, F)_s$ is a fiber sub-bundle of $Hom(E \circ E, F)$. In particular it's a submanifold and his condimension is given by $(\star)$, with k the fiber dimension of E and l the fiber dimension of F.
Let $S_{r,s}$ be the pre-image under the map $(\bullet)$ of $Hom(K \circ K, L)_s$. Since this map is a submersion then $S_{r,s}$ is a submanifold of $S_r^{(2)}$ of the same codimension of $Hom(K \circ K, L)_s$.

Proposition 10. *Let $f : X \to Y$ be $1-$generic. Then*

$$x \in S_{r,s}(f) \Leftrightarrow j^2 f(x) \in S_{r,s}.$$

Proof. Let x be in $S_r(f)$ and $j^1 f(x) = \sigma \in S_r$. Since $S_r \cong L^r(TX, TY)$ then we have seen that

$$N_\sigma = \frac{T_\sigma J^1(X,Y)}{T_\sigma S_r} = Hom(K_\sigma, L_\sigma).$$

Consider the map

$$j^1 f : X \to J^1(X, Y).$$

Then we know that the differential $d(j^1 f)_x : T_x X \to T_\sigma J^1(X, Y)$ induces a map

$$D(df)_x : T_x X \to Hom(K_\sigma, L_\sigma)$$

which is the intrinsic derivative of (df) at x.
Since $j^1 f \pitchfork S_r$ then $D(df)_x$ is surjective and $kerD(df)_x = T_x S_r(f)$. Now suppose $x \in S_{r,s}$. Then $K_\sigma = ker(df)_x$ intersects $T_x S_r(f) = kerD(df)_x$ in an s-dimensional space, this implies that

$$D(df)_x : K_\sigma \to Hom(K_\sigma, L_sigma)$$

has a kernel of dimension s. Hence $D(df)_x \in Hom(K_\sigma \circ K_\sigma, L_\sigma)_s$ and $j^2 f(x) \in S_{r,s}$.
The converse is equally easy to see. □

Corollary 1. *Let $f : X \to Y$ be smooth. If $j^2 f \pitchfork S_{r,s}$ then $S_{r,s}(f)$ is a submanifold of $S_r(f)$ whose codimension is given by $(\star)$ where $l = dimY - dimX + k$ and $k = r + max(dimX - dimY, 0)$.*

Thus $S_{r,s}(f)$ is a submanifold of X and $codimS_{r,s} = r^2 + er+$(codim $S_{r,s}(f)$ in $S_r(f)$), where $e = |dimX - dimY|$.
The Transversality Theorem says that te condition $j^2 f \pitchfork S_{r,s}$ is satisfied by a residual set of smooth mappings. A mapping for which this condition is satisfied for all r, s is called $2-$generic.

3.5. **The Whitney Theorem for Generic Mappings between 2-Manifolds.** Let X and Y be $2-$dimensional manifolds and let $f : X \to Y$ be $1-$generic mapping. Then $S_1(f)$ has codimension 1 in X and $S_2(f)$ doesn't occur since it would have to be of codimension 4. Let $p \in S_1(f)$, $q = f(p)$. One of the following two situations can occur

(1) $T_pS_1(f) \oplus ker(df)_p = T_pX$,

(2) $T_pS_1(f) = ker(df)_p$.

Definition 8. Let $f : X \to Y$ s.t. $j^1 f \pitchfork S_1$. Then $x \in S_1(f)$ is a *fold point* if $T_xS_1(f) \oplus ker(df)_x = T_xX$. Otherwise if$T_xS_1(f) = ker(df)_x$ then x is said to be a *cusp point*.

The first theorem of Whitney for maps between $2-$manifolds gives the normal form for fold points.

Theorem 9. *If p is a fold point then one can choose a system of coordinates (x_1, x_2) centered at p and (y_1, y_2) centered at $f(p)$ such that f is the mapping*

$$(x_1, x_2) \mapsto (x_1, x_2^2)$$

Now we will suppose p to be a cusp point, this situation is more complicated. Let us choose a smooth nonvanishing vector field ξ along $S_1(f)$ such that at each point of $S_1(f)$ ξ is in the kernel of (df). By assumption $\xi(p) \in T_pS_1(f)$. Now let k be a smooth function on X such that $k = 0$ on $S_1(f)$ and $(dk)_p \neq 0$. Consider $(dk)(\xi) : S_1(f) \to TX$. Then by assumption $(dk)_p(\xi(p)) = 0$ and one can check that the order of this zero does not depend on the choice of ξ and k.

Definition 9. p is *simple cusp point* if this zero is a simple zero.

The second main theorem of Whitney states

Theorem 10. *If p is a simple cusp then one can find coordinates (x_1, x_2) centered at p and (y_1, y_2) centered at $f(p)$ s.t. f has the form*

$$(x_1, x_2) \mapsto (x_1, x_1x_2 + x_2^3).$$

Now let $f : X \to Y$ be a $2-$generic map.
Then the points in $S_{1,0}(f)$ are folds and those in $S_{1,1}(f)$ are cusps. Moreover from the previous section we have that $dimS_{1,1} = 0$(note that this implies also that cusp points are isolated).
Let x be a cusp point and suppose that x is not a simple cusp. Then it can be proved that there exists coordinates (x_1, x_2) centered at x, (y_1, y_2) centered at $f(x)$ and a map

$$\phi : (x_1, x_2) \mapsto (x_1, x_1\alpha(x_1, x_2))$$

with the same $3-$jet at 0 as f. Then for this map $S_{1,1}(\phi)$ is the whole x_2 axis and since ϕ has the same $3-$jet as f at 0 then we can conclude that $j^2\phi \pitchfork S_{1,1}$ at 0. This implies that locally $S_{1,1}(\phi)$ is a submanifold and $dim S_{1,1}(\phi) = 0$ which is absurd since it is the x_2 axis. Thus if f is $2-$generic the all cusps are simple. We have proved the following

Theorem 11. *Let X and Y be $2-$manifolds. Then there is a residual set in $C^\infty(X, Y)$ s.t. if f belongs to this set, its only singularities are folds and simple cusps.*

3.6. **The Thom-Boardman stratification.** Now it is clear how to define high order version of singularities. If $f : X \to Y$ is $2-$generic, $S_{i,j,k}(f)$ is the set of points of $S_{i,j}$ where the map

$$f : S_{i,j}(f) \to Y$$

drops rank by k. If, by chance, $S_{i,j,k}$ turns out to be a manifold, we can define $S_{i,j,k,l}$ similarly. Thom conjectured that for a residual set of mappings this process could be continued indefinitely. This conjecture was proved by Boardman.

Theorem 12. *For every sequence of integers $r_1, \cdots, r_k$ such that $r_1 + max(0, dimX - dimY) \geq r_2 \geq \cdots \geq r_k \geq 0$, one can define a fiber subbundle, $S_{r_1,\ldots,r_k}$ of $J^k(X, Y)$ such that if $j^1 f$ is transversal to all manifolds $S_{t_1,\ldots,t_l}$, $l < k$, then $S_{r_1,\ldots,r_k}(f)$ is well defined and*

$$x \in S_{r_1,\ldots,r_k}(f) \Leftrightarrow j^k f(x) \in S_{r_1,\ldots,r_k}.$$

Definition 10. We will call a mapping $f : X \to Y$ s.t. $j^k f$ is transversal to all the $S_{r_1,\ldots,r_k}$'s a *Boardman map*.

By Thom's Transversality Theorem it's clear that Boardman maps are residual in $\mathcal{C}^\infty(X, Y)$. If f is a Boardman map then we can partition X into a disjoint union of subsets consinting of the non-singular points $X \setminus \bigcup_{i\neq 0} S_i(f)$ and the Boardman sets $S_{r_1,\ldots,r_k}(f)$ with $r_k = 0$. Then the map

$$f : X \setminus \bigcup_{i\neq 0} S_i(f) \to Y$$

is an immersion or a submersion depending on $dimX$ and $dimY$; and if $r_k = 0$ the map

$$f : S_{r_1,\ldots,r_k}(f) \to Y$$

is an immersion. This partition of X is called *Thom-Boardman stratification* and has the property that f, restricted to each stratum, is a particulary simple kind of map (either an immersion or a submersion).

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