The 19th Hilbert's problem
The solution in the two-dimensional case

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Abstract

The XIX Hilbert's problem deals with regularity of (local) minimizers in the variational framework. Ennio De Giorgi and John Forbes Nash solved it independently in 1956-1957 and 1957-1958 respectively, using different methods. The core of the problem was to prove that weak solutions of the Euler-Lagrange equations associated with a given energy functional are indeed classical solutions with hölder continuous gradient, and gain regularity from the coefficients. The general proof is rather technical, but in $\mathbb{R}^2$ an easy and elegant argument based on the direct use of Campanato's criterion (see e.g. [5]) is available.
After a brief discussion about weak solutions and Euler-Lagrange equations, based on [1] and [2], the Hilbert problem will be discussed, trying to avoid technicalities, following mostly [3]. It will be shown that the gradient of the minimizer solves a specific PDE; then, according to the theory of Campanato spaces, we will find out that the gradient is also hölder continuous. In doing this, we will use, without proving them, a large amount of results from general regularity theory for systems of PDEs and from Schauder's theory, see e.g. [4].

References

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