Esercizio 1:

$$z^4 + \left(\frac{4}{1+i} + 2i\right) z = 0 \qquad z\left(z^3 + \left(\frac{\overset{2-2i}{\cancel{4-4i}}}{\cancel{2}} + \cancel{2i}\right)\right) = z(z^3+2) = 0$$

Prima soluzione $\boxed{z=0}$

Se $z \neq 0$ allora

$$z^3 = -2 \qquad z^3 = 2e^{i\pi} \qquad z = \rho e^{i\theta}$$

$$\rho^3 e^{i3\theta} = 2e^{i\pi} \Rightarrow \begin{cases} \rho^3 = 2 \\ 3\theta = \pi + 2k\pi \end{cases} \qquad \begin{array}{l} \rho = \sqrt[3]{2} \\ \theta = \dfrac{\pi + 2k\pi}{3} \end{array}$$

$\boxed{\text{Soluzioni: } z_1 = 0 \quad z_2 = \sqrt[3]{2}\, e^{i\pi/3} \quad z_3 = \sqrt[3]{2}\, e^{i\pi} = -\sqrt[3]{2} \quad z_4 = \sqrt[3]{2}\, e^{i\frac{5}{3}\pi}}$

4 soluzioni distinte

Esercizio 2:

a)
$$U: \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_4 + x_5 = 0 \\ 2x_2 + 2x_4 + 2x_5 = 0 \end{cases} \qquad \begin{cases} x_1 = x_3 \\ x_2 = -x_4 - x_5 \end{cases} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad \begin{pmatrix} x_3 \\ -x_4 - x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\boxed{B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \dim U = 3} \qquad \boxed{B_W = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \dim W = 2}$$

$$w \in W \quad w = \begin{pmatrix} a \\ -a-b \\ b \\ b \\ b \end{pmatrix} \qquad w \in U \iff \begin{cases} a - b = 0 \\ -a - \cancel{b} + \cancel{b} + b = 0 \end{cases} \qquad \begin{cases} a = b \\ a = b \end{cases}$$

$$U \cap W = \left\{ \begin{pmatrix} a \\ -2a \\ a \\ a \\ a \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$\boxed{B_{U\cap W} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \dim U \cap W = 1}$

U e W non sono in somma diretta

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 3 + 2 - 1 = 4$$

$$U + W = \left\langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} \right\rangle$$

sono lin. indipendenti perché $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \notin U$ in quanto $x_1 = 1 \neq x_3 = 0$

ma $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in W \subseteq U + W$

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$$\Rightarrow \boxed{B_{U+W} = \left\{ \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix} \right\} \quad \dim(U+W) = 4}$$

In alternativa

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \end{array} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow B'_{U+W} = \left\{ \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\\1\end{pmatrix} \right\}$$

b) Essendo $B_U = \left\{ \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\0\\1\end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

come T possiamo prendere $T = \langle e_3, e_5 \rangle$

in questo modo $\dim(U+T) = 5$ ed essendo $\dim T = 2$ $\dim U = 3 \Rightarrow T \cap U = \{\bar{0}\}$

$$\boxed{T = \langle e_3, e_5 \rangle = \left\langle \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\rangle}$$

Come visto in a) $\dim(U+W) = 4$ $\dim(U) = 3 \Rightarrow \dim S = 1$

$S = \langle w \rangle$ con $w \in W \setminus U$ $w = \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}$ soddisfa le richieste

$$\boxed{S = \left\langle \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix} \right\rangle}$$

c) $\ker P_W = W^\perp$: $\begin{cases} x_1 - x_2 = 0 \\ -x_2 + x_3 + x_4 + x_5 = 0 \end{cases}$ $\begin{cases} x_1 = x_2 = x_3 + x_4 + x_5 \\ x_2 = x_3 + x_4 + x_5 \end{cases}$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$

$$W^\perp = \left\{ \begin{pmatrix} x_3 + x_4 + x_5 \\ x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \middle| \; x_3, x_4, x_5 \in \mathbb{R} \right\}$$

$$\boxed{B_{\ker P_W} = \left\{ \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\1\end{pmatrix} \right\} \quad \dim \ker P_W = 3}$$

$$\boxed{\text{No, non esiste } v \in \mathbb{R}^5 : P_W(v) = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \text{ perché } \operatorname{Im} P_W = W \text{ e } \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \notin W.}$$

d) $\begin{pmatrix}2+t\\-4t\\1\\t\\1\end{pmatrix} \in W \iff \begin{pmatrix}2+t\\-4t\\1\\t\\1\end{pmatrix} = \begin{pmatrix}a\\-a-b\\b\\b\\b\end{pmatrix}$ $\quad \begin{cases}2+t=a\\-4t=-a-b\\1=b\\t=b\\\cancel{1=b}\end{cases}$ $\quad \begin{cases}2+1=a\\-4=-a-1\\b=1\\t=1\end{cases}$ $\quad \begin{cases}a=3\\a=4-1=3\\b=1\\t=1\end{cases}$

$\boxed{\text{Soluzione solo per } t=1 \quad \begin{pmatrix}3\\-4\\1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-3-1\\1\\1\\1\end{pmatrix} = 3\begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\-1\\1\\1\\1\end{pmatrix} \in W}$

Esercizio 3:

$$A_a = \begin{pmatrix}1 & 2a-2 & a+1\\0 & 2a-1 & -1\\0 & 2a^2-2a & -a\end{pmatrix}$$

i) $\det A_a = 1\,(\cancel{-2a^2}+a+\cancel{2a^2}-2a) = -a$

$\boxed{\begin{array}{l}\text{Per } a \in \mathbb{R}\setminus\{0\} \ \det A_a \neq 0 \Rightarrow f_a \text{ è invertibile quindi iniettiva, suriettiva e biiettiva}\\ \text{Per } a=0 \ f_a \text{ non è iniettiva non è suriettiva non è biiettiva}\end{array}}$

Per $a=0$ $\quad A_0 = \begin{pmatrix}1 & -2 & 1\\0 & -1 & -1\\0 & 0 & 0\end{pmatrix}$ $\quad \operatorname{Ker} f_0 : \begin{cases}x-2y+z=0\\-y-z=0\end{cases}$ $\quad \begin{cases}x=2y-z=-2z-z=-3z\\y=-z\end{cases}$ $\quad \begin{pmatrix}-3z\\-z\\z\end{pmatrix}$

$\boxed{\begin{array}{l}\operatorname{Ker} f_0 = \left\langle \begin{pmatrix}-3\\-1\\1\end{pmatrix} \right\rangle \quad B_{\operatorname{Ker} f_0} = \left\{ \begin{pmatrix}-3\\-1\\1\end{pmatrix} \right\} \quad \dim \operatorname{Ker} f_0 = 1 \Rightarrow \dim \operatorname{Im} f_0 = 3-1=2\\ \operatorname{Im} f_0 = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \quad B_{\operatorname{Im} f_0} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}\end{array}}$

ii) $P_{A_a}(x) = \det \begin{pmatrix}1-x & 2a-2 & a+1\\0 & 2a-1-x & -1\\0 & 2a^2-2a & -a-x\end{pmatrix} =$

$= (1-x)\left[-\cancel{2a^2}+a+ax-2ax+x+x^2+\cancel{2a^2}-2a\right] = (1-x)(x^2-ax+x-a)$

Autovalori $\quad 1-x=0 \Rightarrow d_1=1$

$x^2-(a-1)x-a=0 \Rightarrow x = \dfrac{a-1\pm\sqrt{a^2-2a+1+4a}}{2} = \dfrac{a-1\pm(a+1)}{2} = \begin{cases}a\\-1\end{cases}$

$d_2 = a$

$$x^2-(a-1)x-a=0 \Rightarrow x=\frac{a-1\pm\sqrt{a^2-2a+1+4a}}{2}=\frac{a-1\pm(a+1)}{2}= \begin{matrix} \nearrow \\ \searrow -1 \end{matrix}$$

$d_2=a$

$d_3=-1$

Se $a\in\mathbb{R}\setminus\{1,-1\}$ gli autovalori sono
$d_1=1 \quad m_a(1)=1=m_g(1)$
$d_2=a \quad m_a(a)=1=m_g(a)$
$d_3=-1 \quad m_a(-1)=1=m_g(-1)$

ed A_a è diagonalizzabile.

Se $a=1$ gli autovalori sono
$d_1=1 \quad m_a(1)=2$
$d_2=-1 \quad m_a(-1)=1=m_g(-1)$

Se $a=-1$ gli autovalori sono
$d_1=1 \quad m_a(1)=1=m_g(1)$
$d_2=-1 \quad m_a(-1)=2$

iii) Per quanto visto in ii) se $a\in\mathbb{R}\setminus\{1,-1\}$ A_a è diagonalizzabile

Se $a=1$ $A_1=\begin{pmatrix}1&0&2\\0&1&-1\\0&0&-1\end{pmatrix}$ $m_g(1)=\dim\operatorname{Ker}(A_1-\mathbb{I}_3)=3-rg\begin{pmatrix}0&0&2\\0&0&-1\\0&0&-2\end{pmatrix}=3-1=2$

quindi essendo $m_a(1)=m_g(1)=2$ e $m_a(-1)=m_g(-1)=1 \Rightarrow A_1$ è diagonalizzabile.

Se $a=-1$ $A_{-1}=\begin{pmatrix}1&-4&0\\0&-3&-1\\0&4&1\end{pmatrix}$ $m_g(-1)=\dim\operatorname{Ker}(A_{-1}+\mathbb{I}_3)=3-rg\begin{pmatrix}2&-4&0\\0&-2&-1\\0&4&2\end{pmatrix}=3-2=1$

quindi essendo $m_a(-1)=2\neq m_g(-1)=1 \Rightarrow A_{-1}$ non è diagonalizzabile.

Quindi $\boxed{A_a \text{ è diagonalizzabile se e solo se } a\in\mathbb{R}\setminus\{-1\}}$

iv) $A_1=\begin{pmatrix}1&0&2\\0&1&-1\\0&0&-1\end{pmatrix}$ autovalori $1 \quad m_a(1)=m_g(1)=2$
$-1 \quad m_a(-1)=m_g(-1)=1$
$\Rightarrow D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

$V_1=\operatorname{Ker}(A_1-\mathbb{I}_3)=\operatorname{Ker}\begin{pmatrix}0&0&2\\0&0&-1\\0&0&-2\end{pmatrix}=\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle$

$z=0$

$V_{-1}=\operatorname{Ker}(A_1+\mathbb{I}_3)=\operatorname{Ker}\begin{pmatrix}2&0&2\\0&2&-1\\0&0&0\end{pmatrix}$ $\begin{cases}2x+2z=0\\2y-z=0\end{cases}$ $\begin{cases}x=-z=-2y\\z=2y\end{cases}$ $\begin{pmatrix}-2y\\y\\2y\end{pmatrix}$

$V_{-1} = \langle \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \rangle$

$$\boxed{H = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

D) f_a è simmetria se e solo se $f_a^2 = id_{\mathbb{R}^3} \iff A_a$ è diagonalizzabile con unici autovalori 1 e -1, $\Rightarrow$ solo per $a = 1$, perché se $a \in \mathbb{R} \setminus \{1,-1\}$ a è autovalore diverso da 1 e -1 quindi f_a non può essere una simmetria.
Se $a=-1$ f_a non è diagonalizzabile quindi non può essere una simmetria.
Se $a=1$ f_a è simmetria di asse V_1 e direzione V_{-1}.

Esercizio 41

$P = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \qquad r: \begin{cases} x+2y = -5 \\ z = 1 \end{cases}$

1) $\begin{cases} x = -5-2y \\ z = 1 \end{cases} \qquad \begin{pmatrix} -5-2y \\ y \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} + \langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \rangle}$

Piano $\perp$ a r passante per P $\quad -2x+y = 2-2 = 0 \qquad P' \begin{cases} x+2y=-5 \\ z=1 \\ -2x+y=0 \end{cases}$

$\begin{cases} x+4x=-5 \\ z=1 \\ y=2x \end{cases} \quad \begin{cases} x=-1 \\ z=1 \\ y=-2 \end{cases} \qquad P' = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \qquad d(P,r) = d(P,P') = \|P'-P\| = \left\| \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \right\| = 2$

$\boxed{d(P,r) = 2}$

2) $r \subseteq \pi$, $O \in \pi \qquad \alpha(x+2y+5) + \beta(z-1) = 0$

$5\alpha - \beta = 0 \qquad \beta = 5\alpha \quad \begin{cases} \alpha = 1 \\ \beta = 5 \end{cases}$

$\pi: x+2y+5+5z-5=0 \qquad \boxed{\pi: x+2y+5z=0}$

$\pi' \parallel \pi$ e passante per $P = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \qquad \pi': x+2y+5z = -1-4+15 = 10$

$\boxed{\pi': x+2y+5z=10} \qquad d(\pi,\pi') = d(\pi,P) = \dfrac{|-1-4+15|}{\sqrt{1+4+25}} = \dfrac{10}{\sqrt{30}}$

$$\boxed{d(\pi,\pi') = \frac{10}{\sqrt{30}}}$$

3) $S = P \vee O = O + \langle P-O \rangle = \begin{pmatrix}0\\0\\0\end{pmatrix} + \langle \begin{pmatrix}-1\\-2\\3\end{pmatrix} \rangle$ $\quad \begin{cases} x = -a \\ y = -2a \\ z = 3a \end{cases}$ $\quad \begin{cases} a = -x \\ y = 2x \\ z = -3x \end{cases}$

$$\boxed{S: \begin{cases} 2x - y = 0 \\ 3x + z = 0 \end{cases}}$$

Posizione reciproca $\quad s: \begin{pmatrix}0\\0\\0\end{pmatrix} + \langle \begin{pmatrix}-1\\-2\\3\end{pmatrix} \rangle$ $\quad$ certamente non sono

$r: \begin{pmatrix}-5\\0\\1\end{pmatrix} + \langle \begin{pmatrix}-2\\1\\0\end{pmatrix} \rangle$ $\quad$ parallele perchè $\langle \begin{pmatrix}-1\\-2\\3\end{pmatrix} \rangle \neq \langle \begin{pmatrix}-2\\1\\0\end{pmatrix} \rangle$

allora se $\dim(r \vee s) = 2$ sono incidenti

se $\dim(r \vee s) = 3$ sono sghembe

$\dim(r \vee s) = rg \begin{pmatrix} -5 & -1 & -2 \\ 0 & -2 & 1 \\ 1 & 3 & 0 \end{pmatrix} = 3$ perché $\det \begin{pmatrix} -5 & -1 & -2 \\ 0 & -2 & 1 \\ 1 & 3 & 0 \end{pmatrix} = 15 - 5 = 10 \neq 0$

$\boxed{\text{r e S sono sghembe.}}$