Esercizio 1:

$|z|^3 z^2 = 16\bar{z}i$ se $z=0$ $|z|=0$ $\bar{z}=0$ quindi

l'equazione è verificata perciò $\boxed{z=0}$ è la prima soluzione.

Se $z\neq 0$ $\Rightarrow$ scriviamo z in forma esponenziale

$z=\rho e^{i\theta}$ con $\rho=|z|>0$ numero reale

$\bar{z}=\rho e^{-i\theta}$ $i=e^{i\pi/2}$

In forma esponenziale l'equazione $|z|^3z^2=16\bar{z}i$ risulta

$$\rho^3\rho^2 e^{i2\theta}=16\rho e^{-i\theta}e^{i\pi/2}=16\rho e^{i(\pi/2-\theta)}$$

$$\Rightarrow \begin{cases} \rho^5=16\rho \\ 2\theta=\frac{\pi}{2}-\theta+2k\pi \quad k\in\mathbb{Z} \end{cases} \longrightarrow \begin{cases} \rho=\sqrt[4]{16}=2 \\ 3\theta=\frac{\pi}{2}+2k\pi \quad k\in\mathbb{Z} \end{cases}$$

le soluzioni sono $\begin{cases} \rho=2 \\ \theta=\frac{\pi}{6}+\frac{2k\pi}{3} \end{cases}$ per $k=0,1,2$ quindi in tutto

abbiamo $\boxed{4 \text{ soluzioni distinte } z_1=0,\ z_2=2\left(\cos\frac{\pi}{6}+i\,\text{sen}\frac{\pi}{6}\right),\ z_3=2\left(\cos\frac{5}{6}\pi+i\,\text{sen}\frac{5}{6}\pi\right),\ z_4=2\left(\cos\frac{3}{2}\pi+i\,\text{sen}\frac{3}{2}\pi\right)}$

Esercizio 2:

a) In $\mathbb{R}^4$ si consideri il sottospazio

$U: x_2-x_3=0$ $U=\left\{\begin{pmatrix}x_1\\x_3\\x_3\\x_4\end{pmatrix} \middle|\ x_1,x_3,x_4\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\rangle$

$$\boxed{B_U=\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\} \quad \dim U=3}$$

$W=\left\langle\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\rangle$ $\text{rg}\begin{pmatrix}1&1&1&1\\2&-1&-1&2\\0&1&0&0\end{pmatrix}=\text{rg}\begin{pmatrix}1&1&1&1\\0&1&0&0\\0&-3&-3&0\end{pmatrix}=\text{rg}\begin{pmatrix}1&1&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix}=3$

$=\left\langle\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\rangle$

$$\begin{cases} x_1 = a \\ x_2 = a+b \\ x_3 = a+c \\ x_4 = a+ \end{cases} \qquad \begin{cases} x_2 = x_1+b \\ x_3 = x_1+c \\ x_4 = x_1 \end{cases} \qquad \begin{cases} x_3 = x_1+c \\ x_4 = x_1 \end{cases} \qquad W:\ x_1 - x_4 = 0$$

$$\boxed{B_W = \left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\} \quad \dim W = 3 \quad \text{eq. cartesiana: } x_1 - x_4 = 0}$$

$$U \cap W = \begin{cases} x_2 - x_3 = 0 \\ x_1 - x_4 = 0 \end{cases} \qquad U \cap W = \left\{ \begin{pmatrix}x_1\\x_2\\x_2\\x_1\end{pmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \right\rangle$$

$$\boxed{B_{U\cap W} = \left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \right\}, \quad \dim U \cap W = 2}$$

non sono in $\oplus$

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W) = 3+3-2 = 4$$

$$\Rightarrow U+W = \mathbb{R}^4$$

$$\boxed{B_{U+W} = \{e_1, e_2, e_3, e_4\}, \quad \dim(U+W) = 4}$$

b) $U \cap W = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \right\rangle$ $\quad \dim U = 3 \quad$ se $\underset{1\ +\ 2\ =\ 3}{T \oplus (U \cap W) = U}$ $\Rightarrow \dim T = 1$

$\Rightarrow T = \langle \bar{v} \rangle$ con $\bar{v} \in U \setminus (U \cap W)$ ad esempio $\bar{v} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \in U$ $\bar{v} \notin U \cap W$

$\Rightarrow$ $\boxed{T = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle \text{ verifica le richiesta.}}$

$U:\ x_2 - x_3 = 0$

$W:\ x_1 - x_4 = 0$

$\boxed{v = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}} \notin U$ perché $x_2 = 1 \neq x_3 = 0$

$v \notin W$ perché $x_1 = 1 \neq x_4 = 0$.

Se $S \oplus U = S \oplus W = \mathbb{R}^4 \Rightarrow$ essendo $\dim U = \dim W = 3$

$\dim S = 1 \Rightarrow S = \langle v \rangle$ con $v \in \mathbb{R}^4$ $v \notin U$, $v \in \mathbb{R}^4$ $v \notin W$ $\Rightarrow v = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

verifica le richiesta $\Rightarrow$ $\boxed{S = \left\langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \right\rangle}$

c) $U \cap W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$P_{U\cap W}(v) = (v \cdot u_1)\, u_1 + (v \cdot u_2)\, u_2$ con $\{u_1, u_2\}$ base ortonormale di $U \cap W$

$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} x_1+x_4 \\ 0 \\ 0 \\ x_1+x_4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ x_2+x_3 \\ x_2+x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1 + \frac{1}{2}x_4 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_1 + \frac{1}{2}x_4 \end{pmatrix} \Longrightarrow \boxed{A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}}$$

Ogni matrice di proiezione è diagonalizzabile, quindi A è diagonalizzabile.

Nel nostro caso $A = A^t \Rightarrow B = 100A^{21} - 3A^{12} + 8A$

$B^t = 100(A^{21})^t - 3(A^{12})^t + 8A^t = B$ quindi B è diagonalizzabile.

In alternativa ricordiamo che $A^2 = A$ perché è una proiezione quindi

$B = 100A^{21} - 3A^{12} + 8A = 100A - 3A + 8A = 105A$ quindi

essendo $A \sim D \Rightarrow 105A \sim 105D$ perciò anche B è diagonalizzabile.

Esercizio 3:

$$A = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$$

(i) $\det A = -5(40-4) - 2(-10-8) + 4(4+32) =$

$= -5(36) + 36 + 4\cdot(36) = 36(-5+1+4) = 0 \quad \boxed{\det A = 0}$

$\ker A$: $\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$ Riduciamo con Gauss $\begin{pmatrix} 1 & -4 & 1 \\ -5 & 2 & 4 \\ 4 & 2 & -5 \end{pmatrix}$

$\operatorname{Ker} A$: $\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$ Riduciamo con Gauss $\begin{pmatrix} 1 & -4 & 1 \\ -5 & 2 & 4 \\ 4 & 2 & -5 \end{pmatrix}$

$\begin{pmatrix} 1 & -4 & 1 \\ 0 & -18 & 9 \\ 0 & 18 & -9 \end{pmatrix}$ $\begin{pmatrix} 1 & -4 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 - 4x_2 + x_3 = 0 \\ -2x_2 + x_3 = 0 \end{cases}$ $\begin{cases} x_1 = 4x_2 - x_3 = 4x_2 - 2x_2 = 2x_2 \\ x_3 = 2x_2 \end{cases}$

$$\boxed{\operatorname{Ker} A = \left\{ \begin{pmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{pmatrix} \middle| \, x_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle}$$

-9 è autovalore di $A \iff \operatorname{Ker}(A + 9\mathbb{I}_3) \neq \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$, essendo

$$V_{-9} = \operatorname{Ker}(A + 9\mathbb{I}_3) = \operatorname{Ker} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ -2x_1 - 2x_3 \\ x_3 \end{pmatrix} \middle| \, x_1, x_3 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

$2x_1 + x_2 + 2x_3 = 0 \qquad x_2 = -2x_1 - 2x_3$

$$\boxed{-9 \text{ è autovalore e } V_{-9} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle}$$

(ii) Essendo -9 autovalore con $m_g(-9) = 2 = m_a(-9)$ perché A è simmetrica e 0 è autovalore con $m_g(0) = \dim \operatorname{Ker} A = 1 = m_a(0) \Rightarrow \boxed{D = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

$V_{-9} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle$ cerchiamo una base ortonormale con G-S.

$$u_1 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \frac{1}{5}\left[\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 4 \\ -8 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/5 \\ -2/5 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} \qquad w_2 = \frac{1}{5}\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \qquad \|w_2\| = \frac{1}{5}\sqrt{16 + 4 + 25} = \frac{\sqrt{45}}{5} = \frac{3\sqrt{5}}{5} = \frac{3}{\sqrt{5}}$$

$$u_2 = \frac{1}{5}\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \cdot \frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{15}\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{4\sqrt{5}}{15} \\ \frac{-2\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \end{pmatrix}$$

$$\boxed{H = \begin{pmatrix} 1/\sqrt{5} & \frac{4\sqrt{5}}{15} & 2/3 \\ -2/\sqrt{5} & \frac{-2\sqrt{5}}{15} & 1/3 \\ 0 & \frac{\sqrt{5}}{3} & 2/3 \end{pmatrix}}$$

$$V_0 = \left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle \qquad u_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

Esercizio 4:

a) $\left(\begin{array}{ccc|cccc} 1 & -1 & 0 & 1 & -1 & h-3 & 0 \\ 0 & 1 & -1 & 2 & h-3 & 2 & h-1 \\ 0 & 0 & 1 & -1 & -1 & h-5 & -2 \end{array}\right)$ Riduciamo con Gauss

$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & h-5 & 2h-6 & h-3 \\ 0 & 1 & 0 & 1 & h-4 & h-3 & h-3 \\ 0 & 0 & 1 & -1 & -1 & h-5 & -2 \end{array}\right) \Rightarrow \boxed{A_h = \begin{pmatrix} 2 & 1 & -1 \\ h-5 & h-4 & -1 \\ 2h-6 & h-3 & h-5 \\ h-3 & h-3 & -2 \end{pmatrix}}$

b) $\boxed{\phi_h \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = 2\phi_h \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ h-3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2h-6 \\ 0 \end{pmatrix}}$

c) $A_4 = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}$ Riduciamo con Gauss per calcolare $\ker \phi_4$

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x+z=0 \\ y-3z=0 \end{cases}$ $\begin{cases} x=-z \\ y=3z \end{cases}$

$\ker \phi_4 = \left\{ \begin{pmatrix} -z \\ 3z \\ z \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\rangle$

$\boxed{B_{\ker \phi_4} = \left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\}, \dim \ker \phi_4 = 1}$

$\dim \mathbb{R}^3 = \dim \ker \phi_4 + \dim \operatorname{Im} \phi_4 \Rightarrow \dim \operatorname{Im} \phi_4 = 2$

$3 = 1 + 2$

$\boxed{B_{\operatorname{Im} \phi_4} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \dim \operatorname{Im} \phi_4 = 2}$

Esercizio 5:

1) $r: \begin{cases} x+y=2 \\ 3x-z=0 \end{cases}$ $\qquad \pi: x-2y-z=2$

$r \cap \pi \quad \begin{cases} x+y=2 \\ 3x-z=0 \\ x-2y-z=2 \end{cases}$ $\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 3 & 0 & -1 & 0 \\ 1 & -2 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -3 & -1 & -6 \\ 0 & -3 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -3 & -1 & -6 \\ 0 & 0 & 0 & 6 \end{array}\right)$

$rgA = 2 \neq rg(A|b) = 3 \Rightarrow r \cap \pi = \emptyset$ $\quad$ $\boxed{\text{rette e piano paralleli disgiunti.}}$

$r // \pi$

$r: \begin{cases} y = 2-x \\ z = 3x \end{cases}$ $\quad r: \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\rangle$ $\quad R = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \in r$

$$d(r,\pi) = \underset{\substack{\uparrow \\ \text{perché } r//\pi}}{d(R,\pi)} = \frac{|x_0 - 2y_0 - z_0 - 2|}{\sqrt{1^2+2^2+1^2}} = \frac{|0-4-2|}{\sqrt{6}} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

$\boxed{d(r,\pi) = \sqrt{6}}$

Cerco il piano contenente r ed ortogonale a π: $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \right\rangle$

la cui eq. cartesiana è:

$$\det\begin{pmatrix} x & 1 & 1 \\ y-2 & -1 & -2 \\ z & 3 & -1 \end{pmatrix} = 7x + 4y - 8 - z = 0 \qquad 7x+4y-z=8$$

La proiezione ortogonale di r su π è la retta $\boxed{r': \begin{cases} 7x+4y-z=8 \\ x-2y-z=2 \end{cases}}$

2) Se un piano ha distanza positiva da π allora deve essere parallelo a π

$\Rightarrow \pi': x-2y-z=k$ con $k \in \mathbb{R}$

$P = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \in \pi: x-2y-z=2$

$$d(\pi,\pi') = d(P,\pi') = \frac{|2-k|}{\sqrt{6}} = \sqrt{6} \iff |2-k| = 6 \iff \begin{array}{l} 2-k=6 \iff k=-4 \\ \text{oppure} \\ 2-k=-6 \iff k=8 \end{array}$$

$$\Rightarrow \boxed{\begin{array}{l} \pi_1: x-2y-z=-4 \\ \pi_2: x-2y-z=8 \end{array}}$$ sono i piani richiesti.