EQUAZIONI DIFFERENZIALI

1. EQ. DIFF. SEPARABILI: $y' = f(x) \cdot g(y)$

COME RISOLVO? $\frac{y'}{g(y)} = f(x) \rightsquigarrow \int \frac{y'}{g(y)} dx = \int f(x) dx$

$\rightsquigarrow \int \frac{dy}{\cancel{dx}} \cdot \frac{1}{g(y)} \cdot \cancel{dx} = \int f(x) dx \rightsquigarrow \int \frac{1}{g(y)} dy = \int f(x) dx$

CALCOLO GLI INTEGRALI E CONCLUDO!

ES 2: (a) $xy' - y = 0$

$xy' - y = 0 \Leftrightarrow xy' = y \Leftrightarrow y' = \frac{x}{y}$ SEPARABILE!

$\Leftrightarrow \underbrace{y' \cdot y}_{g(y)} = \underbrace{x}_{f(x)}$

$\Leftrightarrow \int y' \cdot y \, dx = \int x \, dx \Leftrightarrow \int \frac{dy}{dx} \cdot y \cdot dx = \int x \, dx$

$\Leftrightarrow \int y \, dy = \int x \, dx \Leftrightarrow \frac{y^2}{2} + c_1 = \frac{x^2}{2} + c_2$

$\Leftrightarrow y^2 + 2c_1 = x^2 + 2c_2$

$$\Leftrightarrow \quad y^2 = x^2 \underbrace{+2c_2-2c_1}_{c} \Leftrightarrow y = x + c$$

$\hookrightarrow$ INFINITE SOL.

ES. 2

(b) $1+3x^2y^2y'=0$

(c) $(1+x^2)y'+xy=0$

(d) $e^{x+y}+y'e^{x-y}=0$

b) $1+3x^2y^2y'=0$

$$3x^2y^2y' = -1$$

$$y^2y' = -\frac{1}{3x^2}$$

$$\int y^2 y' \, dx = \int -\frac{1}{3x^2} \, dx$$

$$\int y^2 \, dy = -\frac{1}{3}\int x^{-2} \, dx$$

$$\frac{y^3}{3} + c_1 = -\frac{1}{3}\left(-\frac{1}{x} + c_2\right)$$

$$\frac{y^3}{3} + c_1 = \frac{1}{3x} - \frac{1}{3}c_2$$

$$y^3 + 3c_1 = \frac{1}{x} - c_2 \qquad -3c_1 - c_2 = c$$

$$y = \sqrt[3]{\frac{1}{x} + c}$$

c) $(1+x^2)y' + xy = 0$

DA AGGIUNGERE SE DOPO È ESCLUSA

$$y' + \frac{x}{1+x^2}y = 0$$. NOTO $y=0$ È SOLUZIONE

$$\frac{y'}{y} = -\frac{x}{1+x^2}$$

$$\int \frac{y'}{y}\,dx = -\int \frac{x}{1+x^2}\,dx$$

$$\int \frac{1}{y}\,dy = -\int \frac{x}{1+x^2}\,dx$$

$$\ln|y| + c_1 = -\frac{1}{2}\ln|1+x^2| + c_2$$

$$\ln|y| = \ln\underbrace{|1+x^2|}_{>0,\ \text{TOLGO IL MOD.}}{}^{-\frac{1}{2}} + c \qquad e^{a+b} = e^a \cdot e^b$$

$$y = \frac{c}{\ }$$

$$y = \frac{c}{\sqrt{1+x^2}}$$

$\hookrightarrow y>0$, TOLGO IL MOD.

$\hookrightarrow$ INCLUDE $y=0$, $[c=0]$
NON LO AGGIUNGO.

d) $e^{x+y} + y' e^{x-y} = 0$

$$y' e^{x-y} = -e^{x+y}$$

$$y' = -\frac{e^{x+y}}{e^{x-y}} \iff y' = -e^{x+y-x+y}$$

$$y' = -e^{2y}$$

$$\frac{y'}{e^{2y}} = -1 \iff \int \frac{y'}{e^{2y}}\,dx = -\int dx$$

$$\iff \int e^{-2y}\,dy = -\int dx$$

$$-\frac{1}{2}e^{-2y} + c_1 = -x + c_2$$

$$e^{-2y} = +2x + c$$

$$-2y = \ln(2x+c)$$

$$y = -\frac{\ln(2x+c)}{2}$$

- EQ. LINEARI DI PRIMO ORDINE OMOGENEE

$$y' + a(x)y = 0$$

SOLUZIONE GENERALE: $y = ke^{-A(x)}$, $A(x) = \int a(x)\,dx$

- EQ. LIN. DI 1° ORDINE NON-OMOGENEE

$$y' + a(x)y = f(x)$$

L'OMOGENEA ASSOCIATA: $y' + a(x)y = 0$ HA SOLUZIONE

$y = ke^{-A(x)}$, $A(x) = \int a(x)\,dx$

CON IL METODO DI VARIAZ. DELLA COSTANTE OTTENGO LA SEGUENTE FORMULA

$$y = \underbrace{ke^{-A(x)}}_{\text{SOL. OMOG. ASSOCIATA}} + \underbrace{\Big(\underbrace{\int f(x)e^{A(x)}dx}_{\text{"}k(x)\text{"}}\Big)e^{-A(x)}}_{\text{SOL. PARTICOLARE}}, \quad A(x) = \int a(x)\,dx$$

ES 2.

(e) $y' = e^x - \frac{y}{x}$

e) $y' = e^x - \frac{y}{x} \Leftrightarrow y' + \frac{1}{x} y = e^x$

OMOG. ASS: $y' + \frac{1}{x} y = 0$

$a(x) = \frac{1}{x}$, $A(x) = \int \frac{1}{x} dx = \ln|x|$

SOL. OMOG. ASS: $y = K e^{-\ln|x|} \Leftrightarrow y = K e^{\ln(|x|^{-1})}$

$K(x)$: $\int e^x \cdot e^{\ln x} dx$ $\qquad y = K \cdot x^{-1}$

$\int e^x \cdot x \, dx = x e^x - e^x$ $\qquad e^x \cdot x - \int e^x \cdot 1 \, dx$

$e^x x - \int e^x dx$

$e^x \cdot x - e^x$

SOL. GENERALE: $K \cdot \frac{1}{x} + (x e^x - e^x) \frac{1}{x}$

$= \frac{K}{x} + e^x - \frac{e^x}{x}$

$= \frac{K}{x} + e^x \left(1 - \frac{1}{x}\right)$

- PROBLEMA DI CAUCHY

HO UN SISTEMA CON UN'EQ. DIFF. ED UNA CONDIZIONE SULLA SOL.
IN QUESTO MODO POSSO TROVARMI UN INSIEME FINITO DI SOL.

ES 3

(a) $\begin{cases} y - xy + x^2y' = 0 \\ y(1) = 1 \end{cases}$

(b) $\begin{cases} y'(1-x^2) + y^2 - 1 = 0 \\ y(0) = 2 \end{cases}$

(c) $\begin{cases} y' = 4x^3\sqrt{1-y^2} \\ y(0) = 0 \end{cases}$

a) $\begin{cases} y - xy + x^2y' = 0 \\ y(1) = 1 \end{cases}$

$y - xy + x^2y' = 0 \Leftrightarrow y(1-x) + x^2y' = 0$

$\Leftrightarrow y' + \frac{(1-x)}{x^2}y = 0$ EQ. LIN. OMOG.

$A(x) = \int \frac{1-x}{x^2}\,dx = -\frac{1}{x} - \ln(x) + c$

$\Rightarrow y = ke^{\frac{1}{x} + \ln(x)}$

PONGO $y(1) = 1 \Rightarrow 1 = ke \Leftrightarrow k = \frac{1}{e}$

$$e$$

SOL. PROBL. DI CAUCHY: $\frac{1}{e}\cdot e^{\frac{1}{x}+\ln|x|} = e^{\frac{1}{x}+\ln|x|-1}$

b) $\begin{cases} y'(1-x^2)+y^2-1=0 \\ y(0)=2 \end{cases}$

$$y'(1-x^2)+y^2-1=0$$

$$y'=\frac{-y^2+1}{1-x^2}$$

$$\frac{y'}{-y^2+1}=\frac{1}{1-x^2} \quad \text{VAR. SEP}$$

$$\int \frac{y'}{-y^2+1}dx=\int\frac{1}{1-x^2}dx$$

$$\int \frac{1}{1-y^2}dy=\int\frac{1}{1-x^2}dx \qquad (1-y^2)=(1+y)(1-y)$$

$$-\cancel{\frac{1}{2}}\ln\left(\left|\frac{y+1}{y-1}\right|\right)+c_1=-\cancel{\frac{1}{2}}\ln\left(\left|\frac{x+1}{x-1}\right|\right)+c_2$$

$$\frac{1}{(1-y^2)}=\frac{A}{1+y}+\frac{B}{1-y}$$

$$\ln(a)-\ln(b)=\ln\left(\frac{a}{b}\right)$$

$$\frac{y+1}{y-1}=\frac{x+1}{x-1}c$$

$$y+1=(y-1)\frac{x+1}{x-1}c$$

$$y+1 = y\,\frac{x+1}{x-1}\cdot c - \frac{x+1}{x-1}\,c$$

$$y - y\left(\frac{x+1}{x-1}\cdot c\right) = -\frac{x+1}{x-1}\,c - 1$$

$$y\left(1 - \frac{x+1}{x-1}\cdot c\right) = -\left(1 + \frac{x+1}{x-1}\,c\right)$$

$$y = -\frac{\left(1 + \frac{x+1}{x-1}\cdot c\right)}{\left(1 - \frac{x+1}{x-1}\cdot c\right)}$$

PONGO $y(0)=2 \Rightarrow 2 = -\dfrac{(1-c)}{1+c}$

$$\Rightarrow 2+2c = -1+c \Rightarrow c=-3$$

$$\Rightarrow y = -\frac{1+\left(\frac{x+1}{x-1}\right)(-3)}{1-\frac{x+1}{x-1}(-3)}$$

c) $\begin{cases} y' = 4x^3\sqrt{1-y^2} \\ y(0)=0 \end{cases}$

$$y' = 4x^3\sqrt{1-y^2}$$

$\dfrac{y'}{\quad} = 4x^3$ VAR. SEP.

$$\frac{y'}{\sqrt{1-y^2}} = 4x^3 \quad \text{VAR. SEP.}$$

$$\int \frac{1}{\sqrt{1-y^2}}\, dy = 4\int x^3\, dx$$

$$\text{ARCSIN}(y) + c_1 = x^4 + c_2 \, dy$$

$$\Rightarrow \quad y = \sin(x^4 + c)$$

PONGO $y(0) = 0 \Rightarrow 0 = \sin(c) \Rightarrow c = 0 + k\pi, \; k \in \mathbb{Z}$

$$\Rightarrow \quad y = \sin(x^4 + k\pi), \; k \in \mathbb{Z}$$

ES. 3

(d) $\begin{cases} e^x - \frac{y}{x} = y' \\ y(1) = -3 \end{cases}$

(e) $\begin{cases} y' = -y + \frac{e^{-x}}{x^2} \\ y(1) = e \end{cases}$

d) $\begin{cases} e^x - \frac{y}{x} = y' \\ y(1) = 3 \end{cases}$

$e^x - \frac{y}{x} = y' \Rightarrow y' + \frac{1}{x} y = e^x$ EQ. LIN. NON OMOG.

FATTA PRIMA! (PUNTO e) ES. 2)

$\Rightarrow$ SOL: $y = \frac{K}{x} + e^x \left(1 - \frac{1}{x}\right)$

PONGO $y(1) = -3 \Rightarrow -3 = K + e(1-1)$

$\Rightarrow K = -3$

SOL: $y = \frac{-3}{x} + e^x \left(1 - \frac{1}{x}\right)$

e) $\begin{cases} y' = -y + \frac{e^{-x}}{x^2} \\ y(1) = e \end{cases}$

$$\underbrace{a(x)}_{=1} \cdot y$$

$y' = -y + \frac{e^{-x}}{x^2} \Leftrightarrow y' + y = \frac{e^{-x}}{x^2}$ EQ. LIN. NON OMOG.

OMOG. ASS.: $y' + y = 0$

$A(x) = \int dx = x + c \Rightarrow$ SOL: $y = K e^{-x}$

$K(x) = \int f(x) \cdot e^{A(x)} dx = \int \frac{e^{-x}}{x^2} \cdot e^x \, dx$

$$\Rightarrow \int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$$

$$\Rightarrow \text{SOL: } k e^{-x} + \left(-\frac{1}{x}\right) e^{-x}$$

$$= e^{-x}\left(k - \frac{1}{x}\right)$$

PONGO $y(1) = e \Rightarrow e = e^{-1}(k-1)$

$$e^2 = k - 1$$

$$k = e^2 + 1$$

$$\Rightarrow y = e^{-x}\left(e^2 + 1 - \frac{1}{x}\right)$$