Esercizio 1:

Si considerino i sottospazi di $M_{2,2}(\mathbb{R})$

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x_1+x_2-x_4=0 \\ x_3+x_4=0 \end{matrix} \right\}$$

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} 2x_1-x_2-x_4=0 \\ -4x_1+2x_2+2x_4=0 \end{matrix} \right.$$

a) $U: \begin{cases} x_1+x_2-x_4=0 \\ x_3+x_4=0 \end{cases} \quad \begin{cases} x_1=-x_2+x_4 \\ x_3=-x_4 \end{cases} \quad \begin{pmatrix} -x_2+x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \quad x_2, x_4 \in \mathbb{R}$

$$U = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle \qquad \boxed{B_U = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} \quad \dim U = 2}$$

$W: \begin{cases} 2x_1-x_2-x_4=0 \\ -4x_1+2x_2+x_4=0 \end{cases} \quad x_2 = 2x_1 - x_4 \quad \begin{pmatrix} x_1 \\ 2x_1-x_4 \\ x_3 \\ x_4 \end{pmatrix} \quad x_1, x_3, x_4 \in \mathbb{R}$

$$\boxed{B_W = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \dim W = 3}$$

Certamente U e W non sono in somma diretta perché

$\dim U + \dim W = 2+3 = 5 > \dim \mathbb{R}^4 = 4 \Rightarrow \dim(U \cap W) \geq 1$

mentre U e W sono in somma diretta se e solo se $\dim(U \cap W) = 0$.

Calcoliamo $U \cap W$:

$$\begin{cases} x_1+x_2-x_4=0 \\ x_3+x_4=0 \\ 2x_1-x_2-x_4=0 \end{cases} \quad \begin{cases} x_1=-x_2+x_4 \\ x_3=-x_4 \\ -2x_2+2x_4-x_2-x_4=0 \end{cases} \quad \begin{cases} x_1 = 2x_2 \\ x_3 = -3x_2 \\ x_4 = 3x_2 \end{cases} \quad \begin{pmatrix} 2x_2 \\ x_2 \\ -3x_2 \\ 3x_2 \end{pmatrix} \quad x_2 \in \mathbb{R}$$

$$\boxed{B_{U\cap W} = \left\{ \begin{pmatrix} 2\\1\\-3\\3 \end{pmatrix} \right\} \qquad \dim(U\cap W) = 1}$$

$\dim(U+W) = \dim U + \dim W - \dim(U\cap W) = 2+3-1 = 4 \Rightarrow U+W = \mathbb{R}^4$

$$\boxed{B_{U+W} = \{e_1, e_2, e_3, e_4\}}$$

b) $B_{U\cap W} = \left\{ \begin{pmatrix} 2\\1\\-3\\3 \end{pmatrix} \right\}$ $\dim W = 3$ dobbiamo cercare $w_1, w_2 \in W$ tali che $\mathcal{C} = \left\{ \begin{pmatrix} 2\\1\\-3\\3 \end{pmatrix}, w_1, w_2 \right\}$ siano linearmente indipendenti. Ad esempio $w_1 = \begin{pmatrix} 1\\2\\0\\0 \end{pmatrix}$ $w_2 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}$ verificano la richiesta quindi

$$\boxed{\mathcal{C} = \left\{ \begin{pmatrix} 2\\1\\-3\\3 \end{pmatrix}, \begin{pmatrix} 1\\2\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \right\}}$$

$v = \begin{pmatrix} 2\\3\\3\\1 \end{pmatrix} \in W$ perché $2x_1 - x_2 - x_4 = 2\cdot 2 - 3 - 1 = 4-4 = 0$

cioè verifica l'equazione cartesiana di W.

Una base di $\mathbb{R}^4$ che contenga v è $\{v, e_2, e_3, e_4\}$.

c) Determinare una base ortonormale di U. Applichiamo il procedimento di G-S a $B_U = \left\{ \begin{pmatrix} -1\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\-1\\1 \end{pmatrix} \right\}$

$$\boxed{u_1 = \begin{pmatrix} -1/\sqrt{2}\\1/\sqrt{2}\\0\\0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1\\1\\0\\0 \end{pmatrix}}$$

$$\begin{pmatrix} 1\\0\\-1\\1 \end{pmatrix} - \left[\begin{pmatrix} 1\\0\\-1\\1 \end{pmatrix} \cdot \begin{pmatrix} -1/\sqrt{2}\\1/\sqrt{2}\\0\\0 \end{pmatrix} \right] \begin{pmatrix} -1/\sqrt{2}\\1/\sqrt{2}\\0\\0 \end{pmatrix} = \begin{pmatrix} 1\\0\\-1\\1 \end{pmatrix} + \begin{pmatrix} -1/2\\1/2\\0\\0 \end{pmatrix} = \begin{pmatrix} 1/2\\1/2\\-1\\1 \end{pmatrix} \qquad \boxed{u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1\\1\\-2\\2 \end{pmatrix}}$$

$$P_U(v') = (v'\cdot u_1)u_1 + (v'\cdot u_2)u_2 = \frac{1}{2}\left[\begin{pmatrix} 0\\1\\-3\\4 \end{pmatrix} \cdot \begin{pmatrix} -1\\1\\0\\0 \end{pmatrix} \right] \begin{pmatrix} -1\\1\\0\\0 \end{pmatrix} + \frac{1}{10}\left[\begin{pmatrix} 0\\1\\-3\\4 \end{pmatrix} \cdot \begin{pmatrix} 1\\1\\-2\\2 \end{pmatrix} \right] \begin{pmatrix} 1\\1\\-2\\2 \end{pmatrix} =$$

$$= \begin{pmatrix} -1/2\\1/2\\0\\0 \end{pmatrix} + \frac{\cancel{15}^{\,3}}{\cancel{10}_{\,2}} \begin{pmatrix} 1\\1\\-2\\2 \end{pmatrix} = \begin{pmatrix} 1\\2\\-3\\3 \end{pmatrix} \qquad \boxed{P_U(v') = \begin{pmatrix} 1\\2\\-3\\3 \end{pmatrix}}$$ come verifica

$$v' = \begin{pmatrix}0\\1\\-3\\4\end{pmatrix} = \begin{pmatrix}1\\2\\-3\\3\end{pmatrix} + \begin{pmatrix}-1\\-1\\0\\1\end{pmatrix} \quad \text{con} \quad \begin{pmatrix}1\\2\\-3\\3\end{pmatrix} \in U \qquad \begin{pmatrix}-1\\-1\\0\\1\end{pmatrix} \in U^\perp$$

Esercizio 2:

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \qquad f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\y+z\\x-z\\2x-y-3z\end{pmatrix} \qquad A = \begin{pmatrix}1&1&0\\0&1&1\\1&0&-1\\2&-1&-3\end{pmatrix}$$

i) Certamente f non è biiettiva perché $\dim \mathbb{R}^3 = 3 \neq 4 = \dim M_{2,2}(\mathbb{R})$

$$\operatorname{Ker} f = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix}x+y=0\\y+z=0\\x-z=0\\2x-y-3z=0\end{matrix} \right\} \qquad \begin{cases}x+y=0\\y+z=0\\x-z=0\\2x-y-3z=0\end{cases} \quad \begin{cases}x=-y\\z=-y\\-y+y=0\\-2y-y+3y=0\end{cases} \quad \begin{cases}x=-y\\z=-y\end{cases}$$

$$\operatorname{Ker} f = \left\{ \begin{pmatrix}-y\\y\\-y\end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}-1\\1\\-1\end{pmatrix} \right\rangle \qquad \boxed{B_{\operatorname{Ker} f} = \left\{ \begin{pmatrix}-1\\1\\-1\end{pmatrix} \right\}} \Rightarrow \underline{f \text{ non è iniettiva}}. \quad \boxed{\dim \operatorname{Ker} f = 1}$$

$$\operatorname{Im} f = \left\langle \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-3\end{pmatrix} \right\rangle$$

dalla formula delle dimensioni

$$\boxed{\dim \operatorname{Im} f = \dim \mathbb{R}^3 - \dim \operatorname{Ker} f = 3 - 1 = 2}$$

$$\boxed{B_{\operatorname{Im} f} = \left\{ \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\0\\-1\end{pmatrix} \right\}} \qquad \text{notiamo che } \begin{pmatrix}0\\1\\-1\\-3\end{pmatrix} = \begin{pmatrix}1\\1\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\1\\2\end{pmatrix}$$

Essendo $\operatorname{Im} f \leq \mathbb{R}^4$ con $\dim \operatorname{Im} f = 2$ e $\dim \mathbb{R}^4 = 4 \Rightarrow \underline{f \text{ non è suriettiva}}$

ii) $\operatorname{rg} A = \dim \operatorname{Im} f = 2$

$$f^{-1}\left\{ \begin{pmatrix}2\\2\\0\\-2\end{pmatrix} \right\} = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix}x+y=2\\y+z=2\\x-z=0\\2x-y-3z=-2\end{matrix} \right\} = \left\{ \begin{pmatrix}2-y\\y\\2-y\end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\} =$$

$$= \begin{pmatrix}2\\0\\2\end{pmatrix} + \left\langle \begin{pmatrix}-1\\1\\-1\end{pmatrix} \right\rangle$$

$$\begin{cases}x+y=2\\y+z=2\\x-z=0\\2x-y-3z=-2\end{cases} \qquad \left(\begin{array}{ccc|c}1&1&0&2\\0&1&1&2\\1&0&-1&0\\2&-1&-3&-2\end{array}\right) \qquad \left(\begin{array}{ccc|c}1&1&0&2\\0&1&1&2\\0&-1&-1&-2\\0&-3&-3&-6\end{array}\right) \qquad \begin{cases}x+y=2\\y+z=2\end{cases} \quad \begin{matrix}x=2-y\\z=2-y\end{matrix}$$

iii) $B = \begin{pmatrix} 2 & -4 & 5 \\ -2 & 4 & 1 \\ 0 & 0 & 12 \end{pmatrix}$ $\quad p_B(x) = \det(B - x\mathbb{I}_3) = \det \begin{pmatrix} 2-x & -4 & 5 \\ -2 & 4-x & 1 \\ 0 & 0 & 12-x \end{pmatrix} =$

$= (2-x)(4-x)(12-x) + 2(-4)(12-x) =$

$= (12-x)(8-6x+x^2-8) = (12-x)(x^2-6x) = \underline{x(12-x)(x-6)}$

Gli autovalori di B sono $\underline{0,\ 12,\ 6}$ tutti con molteplicità algebrica = molteplicità geometrica = 1 $\Rightarrow$ per il criterio di diagonalizzabilità B è diagonalizzabile.

$V_0 = \operatorname{Ker} B = \operatorname{Ker} \begin{pmatrix} 2 & -4 & 5 \\ -2 & 4 & 1 \\ 0 & 0 & 12 \end{pmatrix}$ $\quad \begin{cases} 2x-4y+5z=0 \\ -2x+4y+z=0 \\ 12z=0 \end{cases}$ $\quad \begin{cases} x = 2y \\ z=0 \end{cases}$ $\quad \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix}$

$\underline{V_0 = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle}$

$V_6 = \operatorname{Ker}(B - 6\mathbb{I}_3) = \operatorname{Ker} \begin{pmatrix} -4 & -4 & 5 \\ -2 & -2 & 1 \\ 0 & 0 & 6 \end{pmatrix}$ $\quad \begin{cases} -4x-4y+5z=0 \\ -2x-2y+z=0 \\ 6z=0 \end{cases}$ $\quad \begin{cases} x=-y \\ z=0 \end{cases}$ $\begin{pmatrix} -y \\ y \\ 0 \end{pmatrix}$

$\underline{V_6 = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle}$

$V_{12} = \operatorname{Ker}(B - 12\mathbb{I}_3) = \operatorname{Ker} \begin{pmatrix} -10 & -4 & 5 \\ -2 & -8 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 8 & -1 \\ -10 & -4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 8 & -1 \\ 0 & 36 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} 2x+8y-z=0 \\ y=0 \end{cases}$

$\begin{cases} z=2x \\ y=0 \end{cases}$ $\qquad \underline{V_{12} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle}$

iv) La matrice $C = \begin{pmatrix} 0 & 1 & -4 \\ 0 & 12 & 21 \\ 0 & 0 & 6 \end{pmatrix}$ ha $p_C(x) = -x(12-x)(6-x)$ quindi ha

3 autovalori distinti 0, 6, 12 tutti di molteplicità algebrica = molteplicità geometrica = 1 quindi C è diagonalizzabile ed è simile alla matrice

$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ essendo anche B simile a D per quanto visto in iii) si ha che C è simile a B.

Esercizio 3:

1) $r: \begin{cases} x-y=1 \\ x-z=3 \end{cases}$ $s: \begin{cases} x+(a-1)z=a+4 \\ y-z=a+1 \end{cases}$

$r \cap s_a$ $A|b = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 3 \\ 1 & 0 & a-1 & a+4 \\ 0 & 1 & -1 & a+1 \end{array}\right)$ Riduciamo con Gauss

I° Riga
II°−I°
III°−II°
$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & a & a+1 \\ 0 & 1 & -1 & a+1 \end{array}\right)$ $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & a & a+1 \\ 0 & 0 & 0 & a-1 \end{array}\right)$

Se $a \in \mathbb{R} \setminus \{0,1\}$ $rg A = 3 \neq rg(A|b) = 4$ quindi le rette r e s_a sono sghembe.

Se $a=0$ $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$ $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$rg A = 2 \neq rg(A|b) = 3$ quindi le rette r e s_0 sono parallele distinte.

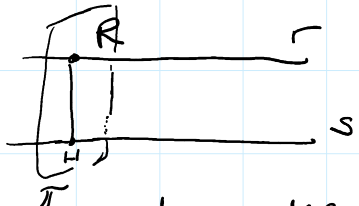
Se $a=1$ $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $rg A = 3 = rg(A|b)$ quindi le rette r e s_1 sono incidenti, risolvendo il sistema si ottiene il punto d'intersezione

$\begin{cases} z=2 \\ y-z=2 \quad y=z+2=4 \\ x-y=1 \quad x=y+1=4+1=5 \end{cases}$ $r \cap s_1 = \left\{ \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} \right\}$

2) Posto $a=0$ $r: \begin{cases} x-y=1 \\ x-z=3 \end{cases}$ $s: \begin{cases} x-z=4 \\ y-z=1 \end{cases}$

$r: \overset{R}{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}} + \langle \overset{v}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \rangle$ $s: \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \langle \overset{v}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \rangle$

Per determinare una coppia di punti di minima distanza considero il piano π ortogonale a r passante per R e determino il punto H intersezione tra π ed s.



(R,H) è coppia di punti di minima distanza e $d(r,s) = d(R,H) = \|H-R\|$

$\pi: \; x+y+z = 1+0-2 = -1$ $\qquad R = \begin{pmatrix}1\\0\\-2\end{pmatrix}$

$\pi \cap s \quad \begin{cases} x+y+z=-1 \\ x-z=4 \\ y-z=1 \end{cases} \quad \begin{cases} z+4+z+1+z=-1 \\ x=z+4 \\ y=z+1 \end{cases} \quad \begin{cases} 3z=-6 \\ x=z+4 \\ y=z+1 \end{cases} \quad \begin{cases} z=-2 \\ x=2 \\ y=-1 \end{cases} \quad H = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$

$d(r,s) = \|H-R\| = \left\| \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\| = \sqrt{2}.$

3) $\quad r: \begin{cases} x-y=1 \\ x-z=3 \end{cases} \qquad s_{-1}: \begin{cases} x-2z=3 \\ y-z=0 \end{cases}$

Piani contenenti s_{-1}: $\sigma: \alpha(x-2z-3)+\beta(y-z)=0$ con $\begin{pmatrix}\alpha\\\beta\end{pmatrix} \in \mathbb{R}^2 \setminus \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$

$V_r = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$ quindi $V_\sigma: \alpha(x-2z)+\beta(y-z)=0$

$\alpha(1-2)+\beta(1-1)=0 \qquad -\alpha=0$

$\sigma: \; y-z=0 \qquad R = \begin{pmatrix}1\\0\\-2\end{pmatrix} \qquad d(r,s_{-1}) = d(R,\sigma) = \frac{|0+2|}{\sqrt{2}} = \sqrt{2}$

4) l'asse è il piano $\perp$ ad $\overline{AB}$ passante per $M = \frac{A+B}{2} = \begin{pmatrix}2\\1\\2\end{pmatrix}$.

$B-A = \begin{pmatrix}2\\0\\2\end{pmatrix} \qquad \tau: \; x+z=4.$