1Appello

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Esercizio 1: Canale A

$$(1+i)x^5 - 16\sqrt{2}x = 0$$

$$x\left(\sqrt{2}\,e^{i\pi/4}x^4 - 16\sqrt{2}\right) = 0 \Rightarrow x = 0 \text{ oppure}$$

$$x^4 = 16\,e^{-i\pi/4} = 16\,e^{i\frac{7}{4}\pi} \qquad \text{essendo } \sqrt[4]{16} = 2$$

$$x = 2\,e^{i\left(\frac{7}{16}\pi + \frac{2k\pi}{4}\right)} \qquad k = 0,1,2,3$$

$$= 2\,e^{i\frac{7\pi + 8k\pi}{16}}$$

Sol:

$$\boxed{\begin{array}{lll} x_0 = 2\,e^{i\frac{7}{16}\pi} & x_2 = 2\,e^{i\frac{23}{16}\pi} & x = 0 \\ x_1 = 2\,e^{i\frac{15}{16}\pi} & x_3 = 2\,e^{i\frac{31}{16}\pi} & \end{array}}$$

Esercizio 2:

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 - x_3 + x_4 = 0 \\ x_1 - x_2 = 0 \\ 3x_1 - 2x_2 - x_3 + x_4 = 0 \end{array} \right\}$$

$$W = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix} \right\rangle$$

a) B_U:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 3 & -2 & -1 & 1 & 0 \end{array}\right) \xrightarrow[\text{III}^\circ - 3\text{I}^\circ]{\text{II}^\circ - \text{I}^\circ} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -2 & 0 \end{array}\right) \xrightarrow{\text{III}^\circ - 2\text{II}^\circ} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = x_3 - x_4 \\ x_2 = x_3 - x_4 \end{cases} \qquad \begin{pmatrix} x_3 - x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} \qquad \boxed{B_U = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \dim U = 2}$$

$$W \; \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \begin{pmatrix} 2 & 1 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 2 & 1 & -3 & 0 \end{pmatrix} \qquad \begin{array}{c} w_1 \\ w_2 - w_1 \\ w_3 - w_1 \end{array} \begin{pmatrix} 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{pmatrix}$$

$$w_3 - w_1 + 3w_2 - 3w_1 = \underline{0}$$

$$-4w_1 + 3w_2 + w_3 = \underline{0}$$

$$\boxed{B_W = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix} \right\}} \qquad \boxed{\dim W = 2}.$$

$U \cap W$: $\quad U: \begin{cases} x_1 = x_3 - x_4 \\ x_2 = x_3 - x_4 \end{cases}$

$w \in W \iff w = a\begin{pmatrix}2\\1\\0\\3\end{pmatrix} + b\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}2a\\a\\b\\3a+b\end{pmatrix}$ $\quad w \in U \cap W \iff \begin{cases} 2a = \cancel{b} - 3a - \cancel{b} \\ a = b - 3a - b \end{cases} \iff a = 0$

$\Rightarrow w = \begin{pmatrix}0\\0\\b\\b\end{pmatrix}$, $\quad U \cap W = \langle \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \rangle$ $\quad \boxed{B_{U\cap W} = \left\{ \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}}$ $\quad \boxed{\dim(U \cap W) = 1}$

Per la formula di Grassmann $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W) = 3$

quindi per determinare una base di $U+W$ è sufficiente, per il teorema di completamento a base, aggiungere ad una base di U un vettore $w \in W \setminus U$

$\boxed{B_{U+W} = \left\{ \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\\3\end{pmatrix} \right\}}$ $\quad \boxed{\dim(U+W) = 3}$

b) $W = \langle \begin{pmatrix}2\\1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \rangle$ $\quad \boxed{T = \langle \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \rangle}$ soddisfa $T \oplus W = \mathbb{R}^4$

Essendo $\dim U = 2$ $\dim(U+W) = 3 \Rightarrow \dim S = 1$ dobbiamo cercare

$w \in W$ $w \notin U$ $\quad w = \begin{pmatrix}2\\1\\0\\3\end{pmatrix}$ soddisfa la richiesta $S = \langle \begin{pmatrix}2\\1\\0\\3\end{pmatrix} \rangle$.

c) $Z_k = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \middle| \begin{matrix} x_1 + x_2 = k^2 + k \\ x_1 - k x_3 = 0 \end{matrix} \right\}$ è sottospazio $\iff k^2 + k = 0$

Per $k = 0$ $\quad Z_0 \cap U$ $\quad \begin{cases} x_1 - x_3 + x_4 = 0 \\ x_1 - x_2 = 0 \\ x_1 + x_2 = 0 \\ x_1 = 0 \end{cases}$ $\quad \begin{cases} x_3 = x_4 \\ x_2 = 0 \\ x_1 = 0 \end{cases}$ $\quad \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \in Z_0 \cap U$

$\Rightarrow$ Non sono in somma diretta

Per $k = -1$ $\quad Z_{-1} \cap U$ $\quad \begin{cases} x_1 - x_3 + x_4 = 0 \\ x_1 - x_2 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$ $\quad \begin{cases} x_4 = 0 \\ x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$ $\Rightarrow$ Z_{-1} e U sono in somma diretta

$\boxed{k = -1}$.

Esercizio 3:

Sia $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$f_a\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+(5+a)y+z \\ 5x+25y+5z \\ (1+a)x+5y+z \end{pmatrix} \qquad A_a = \begin{pmatrix} 1 & 5+a & 1 \\ 5 & 25 & 5 \\ 1+a & 5 & 1 \end{pmatrix}$$

Essendo f_a endomorfismo, f_a è biiettivo $\Leftrightarrow$ iniettivo $\Leftrightarrow$ suriettivo $\Leftrightarrow \det A_a \neq 0$

$$\det A_a = \det\begin{pmatrix} 1 & 5+a & 1 \\ 5 & 25 & 5 \\ 1+a & 5 & 1 \end{pmatrix} = \underset{1^a \text{colonna} - 3^a \text{colonna}}{\det}\begin{pmatrix} 0 & 5+a & 1 \\ 0 & 25 & 5 \\ 1 & 5 & 1 \end{pmatrix} = 25+5a-25 = 5a$$

Se $a \neq 0$ $\quad \operatorname{Ker} f_a = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ $\quad \dim \operatorname{Ker} f_a = 0$ $\quad B_{\operatorname{Ker} f_a} = \emptyset$ $\Rightarrow$ $\boxed{\text{è iniettiva suriettiva e biiettiva} \Leftrightarrow a \neq 0}$

$\operatorname{Im} f_a = \mathbb{R}^3$ $\quad \dim \operatorname{Im} f_a = 3$ $\quad B_{\operatorname{Im} f_a} = \{e_1, e_2, e_3\}$

Se $a=0$ $\quad A_0 = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 25 & 5 \\ 1 & 5 & 1 \end{pmatrix}$ $\quad \operatorname{rg} A_0 = \dim \operatorname{Im} f_0 = 1$ $\quad \boxed{B_{\operatorname{Im} f_0} = \left\{\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}\right\}}$

$$\boxed{\operatorname{Ker} f_0 = \left\langle \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \right\rangle}$$

ii) $\boxed{A_a \text{ è ortogon. diagonalizzabile} \Leftrightarrow A_a = A_a^t \Leftrightarrow a=0}$

$$A_0 = \begin{pmatrix} 1 & 5 & 1 \\ 5 & 25 & 5 \\ 1 & 5 & 1 \end{pmatrix} \qquad P_{A_0}(x) = \det\begin{pmatrix} 1-x & 5 & 1 \\ 5 & 25-x & 5 \\ 1 & 5 & 1-x \end{pmatrix} = \det\begin{pmatrix} -x & 5 & 1 \\ 0 & 25-x & 5 \\ x & 5 & 1-x \end{pmatrix} =$$

$$= x \det\begin{pmatrix} -1 & 5 & 1 \\ 0 & 25-x & 5 \\ 1 & 5 & 1-x \end{pmatrix} = x\left[-\cancel{25}+26x-x^2+\cancel{25}+\cancel{25}-\cancel{25}+x\right] =$$

$$= x(-x^2+27x) = -x^2(x-27)$$

Gli autovalori sono 0 $\quad m_a(0) = 2 = m_g(0)$ $\qquad V_0 = \left\langle \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \right\rangle$

$27 \cdot 5$

130

essendo ortogonalm. diagon. $V_{27} = V_0^\perp = \left\langle \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \right\rangle$

$5+26.$

In alternativa si può calcolare

$$V_{27} = \operatorname{Ker}(A_0 - 27 I_3) = \operatorname{Ker}\begin{pmatrix} -26 & 5 & 1 \\ 5 & -2 & 5 \\ 1 & 5 & -26 \end{pmatrix}$$

Riduciamo a scala

$$\begin{array}{l} 3^\circ R \\ 2^\circ - 5\,3^\circ \\ 1^\circ + 26\cdot 3^\circ \end{array}\begin{pmatrix} 1 & 5 & -26 \\ 0 & -27 & 5+26\cdot 5 \\ 0 & 5+26\cdot 5 & 1-26\cdot 26 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -26 \\ 0 & -27 & 5\cdot 27 \\ 0 & 5\cdot 27 & 1-26\cdot 26 \end{pmatrix} \quad \begin{pmatrix} 1 & 5 & -26 \\ 0 & -1 & 5 \\ 0 & 135 & \underset{\substack{\| \\ -135\cdot 5}}{-675} \end{pmatrix} \text{\} sono multipli}$$

$$\Rightarrow V_{27} \begin{cases} x+5y-26z=0 \\ -y+5z=0 \end{cases} \quad \begin{cases} x=-5y+26z=-25z+26z=z \\ y=5z \end{cases} \Rightarrow V_{27} = \left\langle \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \right\rangle$$

$$H = \begin{pmatrix} 5 & 0 & 1 \\ -1 & 1 & 5 \\ 0 & -5 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

iii) Il vettore $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ è autovettore di $A_a \Leftrightarrow$

$A_a v = \alpha v$ esiste $\alpha \in \mathbb{R}$

$$A_a v = \begin{pmatrix} 1 & 5+a & 1 \\ 5 & 25 & 5 \\ 1+a & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7+a \\ 35 \\ 7+a \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{cases} 7+a=\alpha \\ 35=\alpha \\ 7+a=\alpha \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = 35 \\ a = \alpha - 7 = 35-7=28 \end{cases} \qquad \boxed{a=28}$$

iv) $\operatorname{Im} f_0 = \left\langle \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \right\rangle \qquad u = \frac{1}{\sqrt{27}} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$

$$\boxed{P} = \frac{1}{27} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} (1\ 5\ 1) = \boxed{\begin{pmatrix} \frac{1}{27} & \frac{5}{27} & \frac{1}{27} \\ \frac{5}{27} & \frac{25}{27} & \frac{5}{27} \\ \frac{1}{27} & \frac{5}{27} & \frac{1}{27} \end{pmatrix}} = \frac{1}{27} A_0$$

$P^2 = P$ perché è una proiezione quindi $\boxed{P^{32} = P}$.

Esercizio 4:

a) $r: \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle$

Eq. cart. di r $\begin{cases} x = 1+a \\ y = 3-a \\ z = -2+2a \end{cases} \qquad a = x-1 \quad \begin{cases} y = 3-x+1 \\ z = -2+2x-2 \end{cases} \qquad \boxed{\begin{cases} x+y=4 \\ 2x+z=-4 \end{cases}}$

Determinare la retta s passante per $P = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ e ortogonale al piano π: $2x+y+z=8$.

Essendo $V_{\pi}^{\perp} = \langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \rangle$: $s: \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \rangle$

$$\boxed{\begin{cases} x = 1 + 2a \\ y = 3 + a \\ z = -2 + a \end{cases}}$$

b) Determinare le bisettrici delle rette r ed s.

$r = P + \langle v_r \rangle$ $\quad s = P + \langle v_s \rangle$ $\quad$ essendo $v_r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $\quad v_s = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \|v_r\| = \sqrt{6} = \|v_s\|$

$b_1 : P + \langle v_r + v_s \rangle \qquad b_2 : P + \langle v_r - v_s \rangle$

$$\boxed{b_1 : \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \langle \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \rangle \qquad b_2 : \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \langle \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \rangle}$$

c) $d(r, \sigma) =$ $\qquad \sigma : 2x - z = 1$ $\qquad r \parallel \sigma$ perché $V_r = \langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \rangle \subseteq V_\sigma : 2x - z = 0$

$2 \cdot 1 - 2 = 0$

$= d(R, \sigma) = \dfrac{|2 + 2 - 1|}{\sqrt{5}} = \boxed{\dfrac{3}{\sqrt{5}}}$

in altri temi d'esame come il $\oplus$ il piano $\bar{\sigma} : 2x + z = 1$ non è parallelo a $r \Rightarrow r \cap \bar{\sigma} \neq \emptyset \Rightarrow d(r, \bar{\sigma}) = 0$.

d) $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}}$