

# Introduction to Luenberger observers

# Contents map

<u>developed content units</u>	<u>taxonomy levels</u>
observer	u1, e1

<u>prerequisite content units</u>	<u>taxonomy levels</u>
feedback	u1, e1
continuous time LTI systems	u1, e1

## Main ILO of sub-module “Introduction to Luenberger observers”

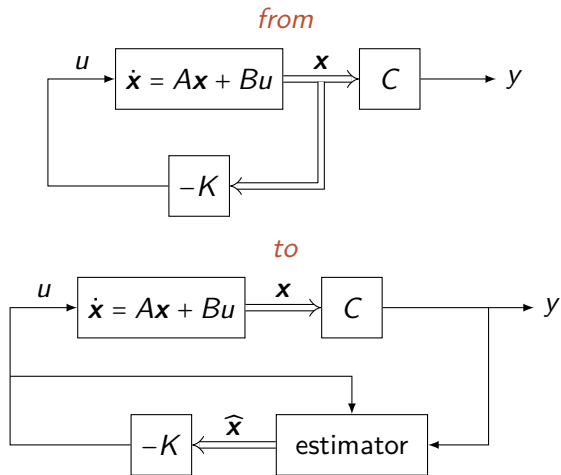
Derive the error dynamics equation for state estimators and explain its significance for observer stability

Describe the meaning of the gain matrix  $L$  in Luenberger observers

List rules of thumb to select estimator poles based on controller dynamics and system characteristics

Discuss the trade-offs between fast and slow observers in the presence of process and measurement noise

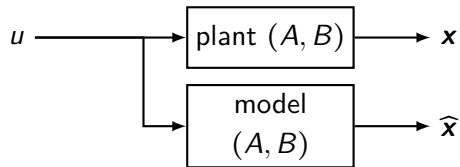
# Control-law design for full-state feedback



## First idea for how to estimate $\mathbf{x}$ : open loop estimator

$$\text{I know } \begin{cases} \mathbf{x}(0) \\ u(k) \\ A, B \end{cases}$$

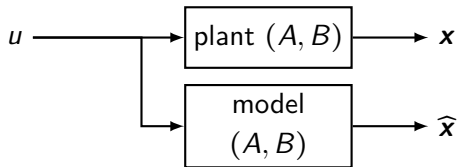
$$\Rightarrow \text{I can simulate } \begin{cases} \hat{\mathbf{x}}(0) = \mathbf{x}(0) \\ \dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu \end{cases}$$



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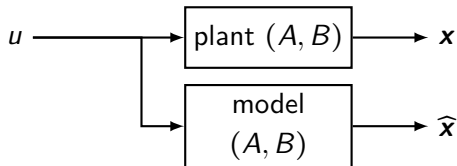
Dynamics:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad \hat{\mathbf{x}}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

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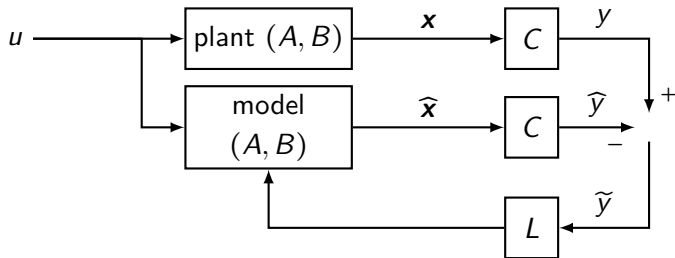
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though, is this strategy robust? *no! if there is uncertainty the estimation error*

$$\tilde{\mathbf{x}}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t) \text{ may diverge}$$

# Estimator design

Idea: use feedback



$$\begin{cases} \hat{\mathbf{x}}(0) = \mathbf{x}(0) \\ \dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L(y - C\hat{\mathbf{x}}) \end{cases}$$



What are the dynamics of the error  $\tilde{\mathbf{x}} := \mathbf{x} - \hat{\mathbf{x}}$ ?

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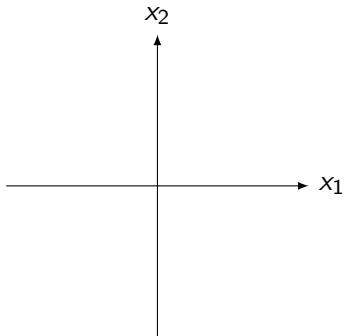
$\Downarrow$

$$\dot{\tilde{\mathbf{x}}} = (A - LC)\tilde{\mathbf{x}}$$

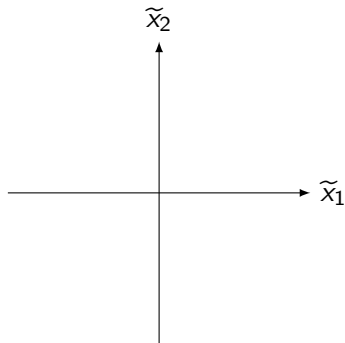
What decides the stability and speed of the dynamics of the error? *The eigenvalues of  $A - LC$ !*

## Dynamics of the error vs. dynamics of the state

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$



$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{x}}$$



## How can we design $L$ ?

For  $(A, B, C, 0)$  fully observable & in observation canonical form:

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad C = [1 \quad 0 \quad \dots \quad 0]$$

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$$L = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{bmatrix} \quad \Rightarrow \quad LC = \begin{bmatrix} L_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ L_n & 0 & \dots & 0 \end{bmatrix}$$



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$\Downarrow$

$$\det(sI - (A - LC)) = (s + a_1 + L_1)(s + a_2 + L_2) \dots (s + a_n + L_n)$$

## Using acker for designing $L$

```
% controller design  
K = acker( A, B, afPoles );  
  
% observer design  
L = ( acker( A', C', afPoles ) )';
```

how to select the poles of the estimator

## Estimator poles selection – rules of thumb

- the observer should be 2 → 6 times faster than the controller

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- the more the sensors are noisy, the slower the observer should be
- the slower the observer, the less resilient the controller is to disturbances

algorithms for designing  $L$ :

- dominant second order poles
- LQR



## Estimator poles selection – connections with estimation theory

$$\begin{cases} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ y &= C\mathbf{x} \end{cases} \mapsto \begin{cases} \dot{\hat{\mathbf{x}}} &= A\hat{\mathbf{x}} + B\mathbf{u} + \mathbf{w} \\ y &= C\hat{\mathbf{x}} + \nu \end{cases}$$

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Estimator as before:

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New error dynamics:

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Trade-off:

- $L$  big  $\implies$  effect of  $\mathbf{w}$  is negligible but  $\nu$  is amplified
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*optimal strategy in a statistical sense requires Kalman filtering* (not in this course)

## Summarizing

Derive the error dynamics equation for state estimators and explain its significance for observer stability

Describe the meaning of the gain matrix  $L$  in Luenberger observers

List rules of thumb to select estimator poles based on controller dynamics and system characteristics

Discuss the trade-offs between fast and slow observers in the presence of process and measurement noise

## Self-assessment activities

## Question 1

Why is an open-loop estimator generally not considered a robust method for state estimation?

### Potential answers:

- I: Because it can track the states accurately even with uncertainties.
- II: Because it uses feedback to correct errors in real time.
- III: Because in the presence of model uncertainties or disturbances, the estimation error may diverge.
- IV: Because it relies on noisy measurements, which destabilize the estimation.
- V: I do not know

## Question 2

What determines the stability and speed of convergence of the estimation error  $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$  in a full-state observer?

### Potential answers:

- I: The eigenvalues of matrix  $B$ .
- II: The input  $u(t)$  and measurement noise.
- III: The initial state  $\mathbf{x}(0)$ .
- IV: The eigenvalues of the matrix  $A - LC$ .
- V: I do not know



## Question 3

In the observer canonical form, how does the gain matrix  $L$  affect the characteristic polynomial of the observer error dynamics?

### Potential answers:

- I: It multiplies the coefficients of the system matrix  $A$ .
- II: It adds to the coefficients of the characteristic polynomial of  $A$ .
- III: It replaces the eigenvalues of  $A$  with the eigenvalues of  $B$ .
- IV: It subtracts from the output matrix  $C$  to reduce measurement noise.
- V: I do not know

## Question 4

Why is it generally recommended for the observer poles to be 2 to 6 times faster than the controller poles?

### Potential answers:

- I: To ensure the controller has enough time to react to the observer.
- II: To make the estimation error dynamics faster than the control system, enabling accurate feedback.
- III: To slow down the observer and reduce noise amplification.
- IV: To match the sampling rate of the digital controller.
- V: I do not know

## Question 5

What is a trade-off involved when choosing faster poles for the observer?

### Potential answers:

- I: Faster poles increase sensitivity to measurement noise.
- II: Faster poles always improve estimation accuracy, regardless of noise.
- III: Slower poles make the observer more responsive.
- IV: Faster poles reduce computational complexity.
- V: I do not know

## Recap of sub-module “Introduction to Luenberger observers”

- one may estimate the states of a system by means of making the estimated state be so that it dynamically matches the measured values
- this strategy though is as valid as the model is, as a description of the system
- the situation is though in practice not as simple as seen here - indeed the here presented case is for “fully observable” systems (a concept that you’ll see in systems theory) and thus not applicable all the times (but extensible to!)

?