The Routh-Hurwitz Stability Criterion

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Main ILO of sub-module "The Routh-Hurwitz Stability Criterion"

Use the Routh-Hurwitz criterion to assess
the stability of a linear time-invariant system

Why all of this?

Routh = algorithm to answer "is this system stable?" without needing to find the poles explicitly

i.e., when having a TF like

$$H(s) = \frac{\mathrm{num}(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

How does it work, in a nutshell?

1. assume to know the characteristic polynomial $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$
2. draw an opportune table, and fill up the first two columns
3. fill up the remaining columns with a simple formula
4. when the table is completed, inspect the first column of that table

Additional resources

Videos:

- https://www.youtube.com/watch?v=WBCZBOB3LCA
- https://www.youtube.com/watch?v=oMmUPvn6lP8

Animations:

- https://www.reddit.com/r/manim/comments/ujrc3a/routh_table_animation_with_manimgl/?rdt=48909
- https://www.muchen.ca/RHCalc/

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n				
s^{n-1}				
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n			
s^{n-1}				
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n			
s^{n-1}	a_{n-1}			
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n	a_{n-2}		
s^{n-1}	a_{n-1}			
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n	a_{n-2}		
s^{n-1}	a_{n-1}	a_{n-3}		
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	
s^{n-1}	a_{n-1}	a_{n-3}		
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 1: draw an opportune table, and fill up the first two columns

assumption: we know $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Example: $s^3 + 5s^2 + 6s + 4$

s^3	1	6
s^2	5	4
s^1		
s^0		

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}				
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1			
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

$$b_1 = \frac{a_n a_{n-3} - a_{n-2} a_{n-1}}{-a_{n-1}}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1			
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2		
s^2			
s^1			
s^0			

$$b_1 = \frac{a_n a_{n-3} - a_{n-2} a_{n-1}}{-a_{n-1}}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2		
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2		
s^2			
s^1			
s^0			

$$b_2 = \frac{a_n a_{n-5} - a_{n-4} a_{n-1}}{-a_{n-1}}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2		
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	
s^2			
s^1			
s^0			

$$b_2 = \frac{a_n a_{n-5} - a_{n-4} a_{n-1}}{-a_{n-1}}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	
s^2			
s^1			
s^0			

$$b_3 = \frac{a_n a_{n-7} - a_{n-6} a_{n-1}}{-a_{n-1}}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2			
s^1			
s^0			

$$b_3 = \frac{a_n a_{n-7} - a_{n-6} a_{n-1}}{-a_{n-1}}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2			
s^1			
s^0			

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2			
s^1			
s^0			

$$c_1 = \frac{a_{n-1}b_2 - a_{n-3}b_1}{-b_1}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2	≈ 1.3		
s^1			
s^0			

$$c_1 = \frac{a_{n-1}b_2 - a_{n-3}b_1}{-b_1}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2	≈ 1.3		
s^1			
s^0			

$$c_2 = \frac{a_{n-1}b_3 - a_{n-5}b_1}{-b_1}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1				
s^0				

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2	≈ 1.3	1	
s^1			
s^0			

$$c_2 = \frac{a_{n-1}b_3 - a_{n-5}b_1}{-b_1}$$

Step 2: fill up the remaining columns with a simple formula

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1	d_1	d_2	d_3	$\cdots$
s^0	e_1	e_2	e_3	$\cdots$

example:

s^5	1	6	3
s^4	5	4	1
s^3	5.2	≈ 2.8	0
s^2	≈ 1.3	1	0
s^1	≈ -1.2	0	0
s^0	1	0	0

Example: $s^3 + 5s^2 + 6s + 4$

s^3	1	6
s^2	5	4
s^1	5.2	0
s^0	4	

Example: $s^3 + 2s^2 - 3s + 4$

s^3	1	−3
s^2	2	4
s^1	−5	0
s^0	4	

Step 3: when the table is completed, inspect the first column of that table

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1	d_1	d_2	d_3	$\cdots$
s^0	e_1	e_2	e_3	$\cdots$

number of sign changes = number of roots with positive real part

Example: $s^3 + 5s^2 + 6s + 4$

s^3	1	6
s^2	5	4
s^1	5.2	0
s^0	4	

no sign changes $\implies$ 0 roots in the right-half plane $\implies$ stable (for continuous time!)

Example: $s^3 + 2s^2 - 3s + 4$

s^3	1	-3
s^2	2	4
s^1	-5	0
s^0	4	

one sign change $\Longrightarrow$ 1 root in the right-half plane $\Longrightarrow$ unstable (for continuous time!)

But what if something like this happens, while I am doing the computations?

s^5	1	1	3
s^4	2	2	5
s^3	0		
s^2			
s^1			
s^0			

$\Longrightarrow$ needs employing some tricks!

Routh's special cases

How many special cases? Actually just two

a zero just in the first column

$$s^3 - 3s + 2$$

a whole row of zeros

$$s^4 + s^3 - 3s^2 - s + 2$$

Special case I: just a zero in the first column

suggested algorithm:

- replace the zero with a small positive number ε, then proceed
- at the end, check the limit as $\varepsilon \to 0$

Example

$$s^3 - 3s + 2 \quad \Longrightarrow \quad \begin{array}{c|cc} s^3 & 1 & -3 \\ s^2 & 0 & 2 \\ s^1 & & \\ s^0 & & \end{array}$$

Example

$$s^3 - 3s + 2 \implies \begin{array}{c|cc} s^3 & 1 & -3 \\ s^2 & 0 & 2 \\ s^1 & & \\ s^0 & & \end{array}$$

using ε instead:

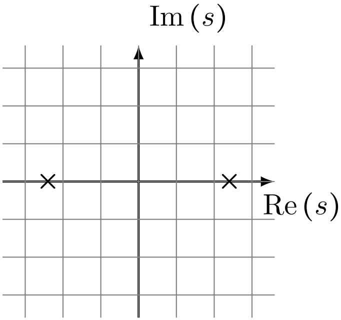
$$\begin{array}{c|cc} s^3 & 1 & -3 \\ s^2 & \varepsilon & 2 \\ s^1 & \dfrac{-3\varepsilon - 2}{\varepsilon} & 0 \\ s^0 & 2 & \end{array}$$

for $\varepsilon \to 0^+$ we have two sign changes $\implies$ unstable

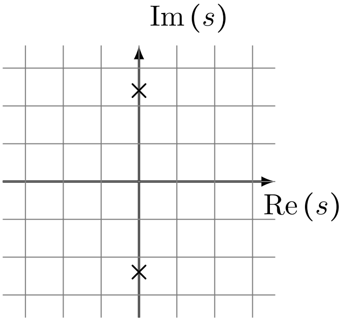
Special case II: an entire row is zero

then one of the following cases may be true:

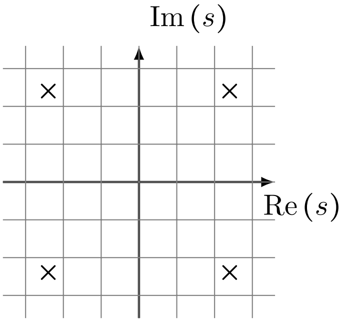
there is a pair of real roots with opposite signs



there is a pair of pure imaginary roots



there are pairs of complex conjugate roots symmetric wrt the origin



Example

s^4	1	-3	2
s^3	1	-1	0
s^2	-2	2	0
s^1	0	0	0
s^0			

Example

$$
\begin{array}{c|ccc}
s^4 & 1 & -3 & 2 \\
s^3 & 1 & -1 & 0 \\
s^2 & -2 & 2 & 0 \\
s^1 & 0 & 0 & 0 \\
s^0 & & &
\end{array}
$$

Algorithm for handling this:

1. form an auxiliary polynomial with coefficients from the row just above the row of zeros
2. take its derivative with respect to s
3. replace the row of zeros with the coefficients of the derivative of the auxiliary equation
4. continue with the Routh array as before (and the number of sign changes will still indicate the number of poles on the RHS of the s-plane)

Step 1: form an auxiliary polynomial with coefficients from the row just above the row of zeros

$$
\begin{array}{c|ccc}
s^4 & 1 & -3 & 2 \\
s^3 & 1 & -1 & 0 \\
s^2 & -2 & 2 & 0 \\
s^1 & 0 & 0 & 0 \\
s^0 & & &
\end{array}
\quad \Longrightarrow \quad A(s) = -2s^2 + 2
$$

Step 2: take its derivative with respect to s

$$
\begin{array}{c|ccc}
s^4 & 1 & -3 & 2 \\
s^3 & 1 & -1 & 0 \\
s^2 & -2 & 2 & 0 \\
s^1 & 0 & 0 & 0 \\
s^0 & & &
\end{array}
\implies A(s) = -2s^2 + 2 \implies \frac{dA(s)}{ds} = -4s
$$

Step 3: replace the row of zeros with the coefficients of the derivative of the auxiliary equation

$$\begin{array}{c|ccc} s^4 & 1 & -3 & 2 \\ s^3 & 1 & -1 & 0 \\ s^2 & -2 & 2 & 0 \\ s^1 & 0 & 0 & 0 \\ s^0 & & & \end{array} \implies A(s) = -2s^2 + 2 \implies \frac{dA(s)}{ds} = -4s$$

$$\implies \text{new table} = \begin{array}{c|ccc} s^4 & 1 & -3 & 2 \\ s^3 & 1 & -1 & 0 \\ s^2 & -2 & 2 & 0 \\ s^1 & 0 & 0 & 0 \\ \text{new:} & -4 & 0 & 0 \\ s^0 & & & \end{array}$$

Step 4: continue with the Routh array as before

s^4	1	-3	2
s^3	1	-1	0
s^2	-2	2	0
s^1	0	0	0
new:	-4	0	0
s^0	2	0	0

four sign changes $\Longrightarrow$ instability (for continuous time!)

Summarizing

Routh-Hurwitz is a powerful test for stability

there are though two special cases you must handle carefully:

- zero in first column: use ε
- row of zeros: construct and differentiate auxiliary polynomial

Most important python code for this sub-module

Check Routh stability with Python (sympy)

```
from sympy import symbols, Matrix, simplify
# define coefficients
coeffs = [1, 2, -3, 4]
# Function to build Routh table
def routh_table(coeffs):
    n = len(coeffs)
    rows = (n + 1) // 2
    table = [[0]*rows for _ in range(n)]
    table[0][:] = coeffs[::2]
    table[1][:len(coeffs[1::2])] = coeffs[1::2]
    for i in range(2, n):
        for j in range(rows - 1):
            num = (table[i-2][0]*table[i-1][j+1] -
                   table[i-2][j+1]*table[i-1][0])
            table[i][j] = simplify(num / table[i-1][0])
    return Matrix(table)
print(routh_table(coeffs))
```

Self-assessment material

Question 1

What does one sign change in the first column of the Routh table indicate?

1. The system is stable.
2. There is one pole on the imaginary axis.
3. There is one pole in the right-half complex plane.
4. I do not know

Solution 1: *Each sign change corresponds to a root with positive real part. One change = one unstable pole.*

Question 2

What does an entire row of zeros in the Routh table suggest?

1. The system is unstable.
2. The system has symmetrical imaginary roots.
3. The system is stable.
4. I do not know

Solution 1: *An entire row of zeros indicates imaginary axis poles and symmetric root pairs, often implying marginal stability.*

Question 3

Stability AnalysisRouth-Hurwitz Criterion What is the primary purpose of the Routh-Hurwitz criterion?

1. To determine the steady-state error of a system
2. To assess the stability of a linear time-invariant (LTI) system without solving for its poles
3. To design a controller for a nonlinear system
4. To compute the frequency response of a system
5. I do not know

Solution 1: *The Routh-Hurwitz criterion is used to determine the stability of an LTI system by analyzing the signs of the first column of the Routh array, without explicitly computing the poles. The correct answer is:* ***To assess the stability of a linear time-invariant (LTI) system without solving for its poles****.*

Question 4

Stability AnalysisRouth-Hurwitz Criterion In the Routh array, what does a sign change in the first column indicate?

1. The system has imaginary poles
2. The system is marginally stable
3. The system has poles in the right-half plane (unstable)
4. The system has a double pole at the origin
5. I do not know

Solution 1: *A sign change in the first column of the Routh array indicates that the system has at least one pole in the right-half plane, making it unstable. The correct answer is:* ***The system has poles in the right-half plane (unstable)****.*

Question 5

Stability AnalysisRouth-Hurwitz Criterion What happens if a row of zeros appears in the Routh array?

1. The system is stable
2. The system has no poles
3. It indicates the presence of symmetric poles (e.g., purely imaginary or real-axis pairs)
4. The Routh-Hurwitz criterion cannot be applied
5. I do not know

Solution 1: *A row of zeros in the Routh array suggests that the system has symmetrically located poles (e.g., purely imaginary or real-axis pairs). An auxiliary polynomial must be constructed to proceed. The correct answer is:* ***It indicates the presence of symmetric poles (e.g., purely imaginary or real-axis pairs)****.*

Question 6

Stability AnalysisRouth-Hurwitz Criterion Which of the following is a necessary condition for applying the Routh-Hurwitz criterion?

1. The system must be nonlinear
2. The characteristic equation must be a polynomial in s
3. The system must have at least one integrator
4. The system must be in state-space form
5. I do not know

Solution 1: *The Routh-Hurwitz criterion requires the characteristic equation of the system to be a polynomial in s. The correct answer is:* ***The characteristic equation must be a polynomial in s.***

Question 7

Stability AnalysisRouth-Hurwitz Criterion If all elements in the first column of the Routh array are positive, what can be concluded?

1. The system is asymptotically stable
2. The system has at least one unstable pole
3. The system is marginally stable
4. The system is nonlinear
5. I do not know

Solution 1: *If all elements in the first column of the Routh array are positive, the system is asymptotically stable (all poles are in the left-half plane). The correct answer is:* ***The system is asymptotically stable****.*

Recap of sub-module "The Routh-Hurwitz Stability Criterion"

1. The Routh-Hurwitz criterion determines the number of unstable poles in a system.
2. It does not require solving for the roots.
3. The number of sign changes in the first column of the Routh table tells us the number of right-half plane poles.

?