Analisi del transitorio per i sistemi del secondo ordine

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Main ILO of sub-module
"Analisi del transitorio per i sistemi del secondo ordine"

Derive the unit step response of a second-order system with complex conjugate poles using Laplace domain analysis

Explain the influence of the natural frequency ω_n and damping ratio ξ on the transient response characteristics (e.g., overshoot, settling time, rise time)

Transform between pole representation and (ω_n, ξ) parametrization for underdamped second-order systems

Estimate transient response features (overshoot, settling time, rise time) using approximate analytical expressions

Classify different step responses based on ξ and ω_n using plotted trajectories and analytic expressions

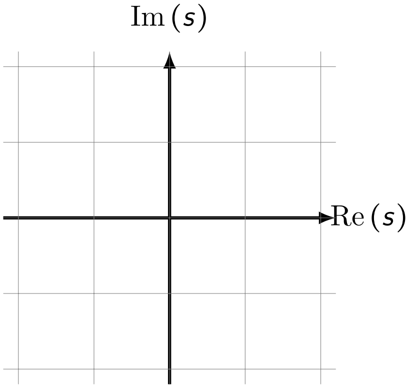
Roadmap

- conti
- interpretazioni grafiche
- espressioni approssimate

Caso piu' complesso: coppia di poli dominanti = singoli e coniugati

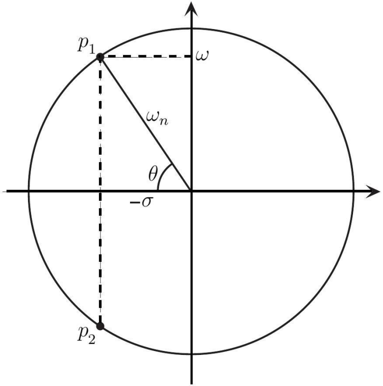
$\Longrightarrow$ approssimazione di $W(s) = \widehat{W}(s) = \dfrac{K}{(s - p_1)(s - p_2)}$

con $W(0) = \widehat{W}(0)$

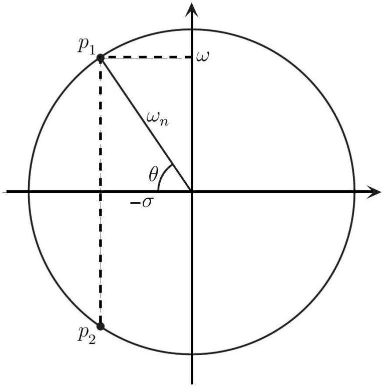


FdT di un sistema del secondo ordine

$$W(s) = \frac{K}{(s-p_1)(s-p_2)} = \frac{K}{(s-p)(s-\overline{p})}$$

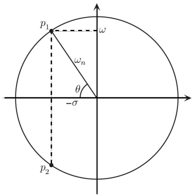


Sistemi del secondo ordine - notazione alternativa



- $\omega_n = |p|$: pulsazione naturale
- $\xi = -\dfrac{\text{Re}[p]}{|p|} > 0$: coefficiente di smorzamento

Come passare tra le notazioni



$$p_{1,2} = -\sigma \pm j\omega \quad \sigma, \omega \geq 0$$

$$\sigma = -\mathrm{Re}[p], \quad \omega = \mathrm{Im}[p]$$

$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\sigma = \xi\omega_n = (\cos\theta)\omega_n$$

$$\omega = \sqrt{1-\xi^2}\omega_n = (\sin\theta)\omega_n$$

Risposta al gradino unitario per sistemi del secondo ordine - conti su come e' la funzione di trasferimento

$$
\begin{aligned}
W(s) &= \frac{K}{(s-p)(s-\overline{p})} \\
&= \frac{K}{s^2-(p+\overline{p})s+p\overline{p}} \\
&= \frac{K}{s^2-2\,\mathrm{Re}[p]s+|p|^2} \\
&= \frac{K}{|p|^2\left(1-2\frac{\mathrm{Re}[p]}{|p|^2}s+\frac{s^2}{|p|^2}\right)} \\
&= \frac{K_B}{1+2\frac{\xi}{\omega_n}s+\frac{s^2}{\omega_n^2}}
\end{aligned}
$$

$$
\omega_n = |p| \quad \xi = -\frac{\mathrm{Re}[p]}{|p|} > 0 \quad K_B = \frac{K}{|p|^2}
$$

Risposta al gradino unitario per sistemi del secondo ordine - conti sulla risposta forzata

$$
\begin{aligned}
Y_f(s) &= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega^2}\frac{1}{s} \\
&= \frac{A}{s} + \frac{B}{s-p} + \frac{C}{s-\overline{p}}
\end{aligned}
$$

$$
A = 1 \quad B = (s-p)Y_f(s)_{|s=p} = \left.\frac{\omega_n^2}{(s-\overline{p})s}\right|_{s=p} = \frac{p\overline{p}}{(p-\overline{p})p} = \frac{\overline{p}}{2j\omega}
$$

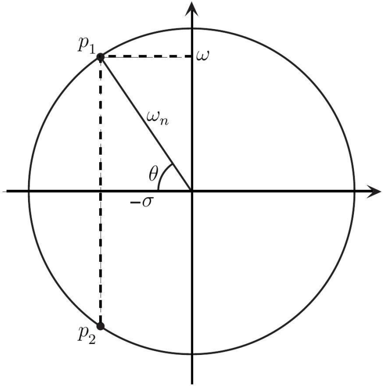
$$
p = -\omega_n e^{-j\theta} \quad B = \frac{-\omega_n e^{j\theta}}{2j\omega} = \frac{-e^{j\theta}}{2j\sin(\theta)} \quad C = \overline{B} = \frac{e^{-j\theta}}{2j\sin(\theta)}
$$

Risposta al gradino unitario per sistemi del secondo ordine - conti sulla risposta forzata

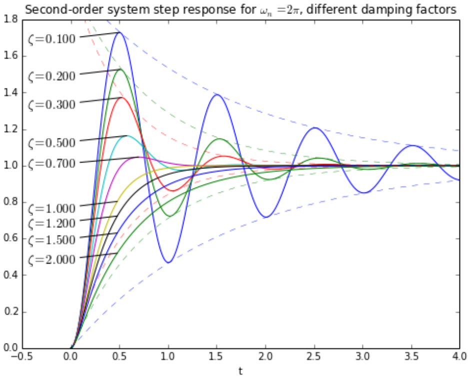
$$
\begin{aligned}
y_f(t) &= 1 - \frac{e^{j\theta}}{2j\sin(\theta)} e^{pt} + \frac{e^{-j\theta}}{2j\sin(\theta)} e^{\overline{p}t} \\
&= 1 - \frac{1}{2j\sin(\theta)} e^{-\sigma t + j\omega t + j\theta} + \frac{1}{2j\sin(\theta)} e^{-\sigma t - j\omega t - j\theta} \\
&= 1 - \frac{e^{-\sigma t}}{\sin(\theta)} \frac{e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}}{2j} \\
&= 1 - \frac{e^{-\sigma t}}{\sin(\theta)} \sin(\omega t + \theta)
\end{aligned}
$$

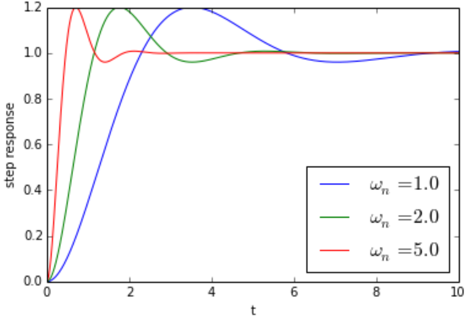
Risposta al gradino unitario per sistemi del secondo ordine - riassunto

$$y_f(t) = 1 - \frac{e^{-\sigma t}}{\sin(\theta)} \sin(\omega t + \theta)$$



Esempi di risposte al gradino unitario per sistemi del secondo ordine





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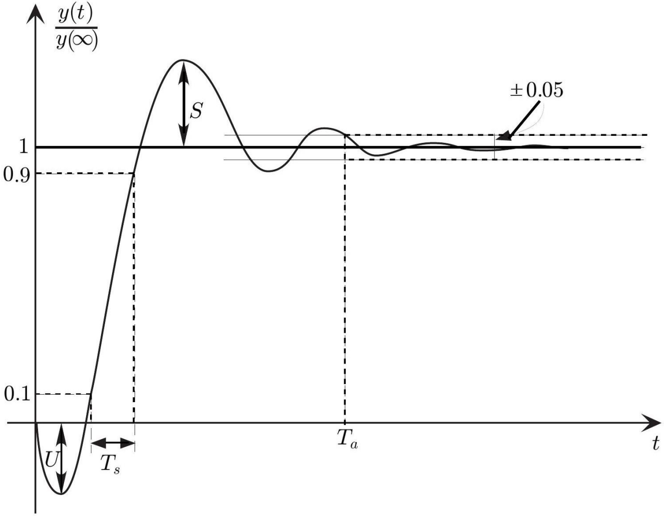
Quantita' interessanti associate a questa risposta



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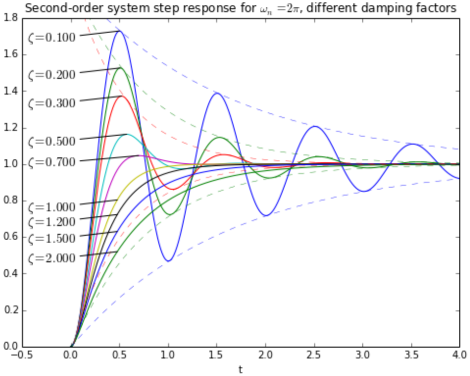
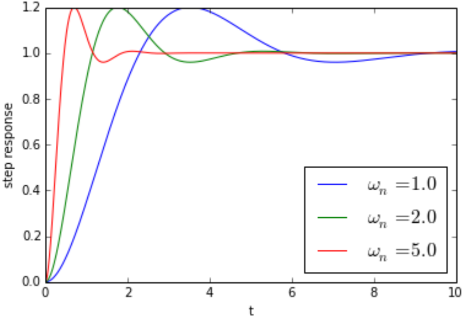


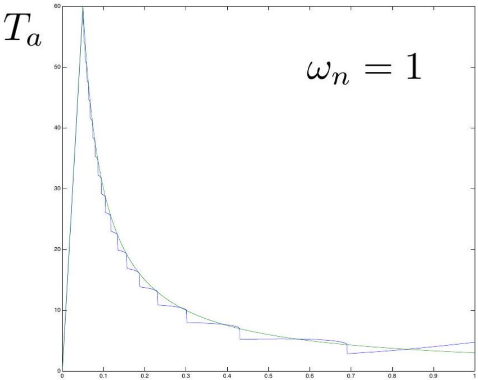


- sottoelongazione $U = 0$
- tempo di assestamento T_a dipende da ξ
- tempo di salita T_s dipende da ω e ξ

Formule associate alle quantita' interessanti

Tempo di assestamento

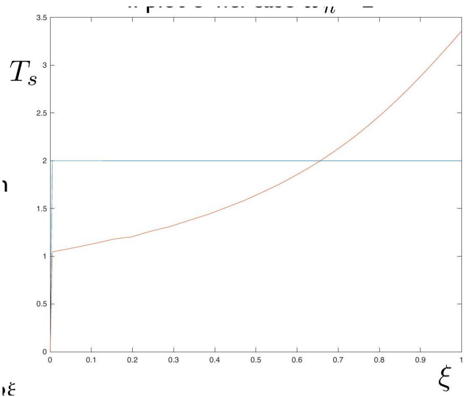
$$T_a \simeq \frac{3}{\xi \omega_n}$$



Formule associate alle quantita' interessanti

Tempo di salita

$$T_s \simeq \frac{2}{\omega_n}$$

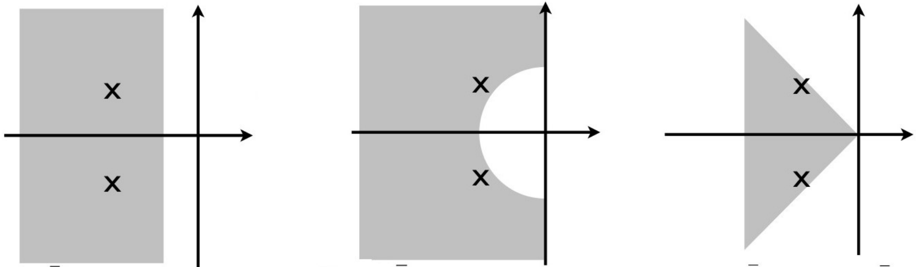


Formule associate alle quantita' interessanti

Sovraelongazione

$$S = \exp\left(-\frac{\delta\pi}{\sqrt{1-\xi^2}}\right)$$

Con questi strumenti gia' possiamo progettare il nostro primo controllore!



$T_a \leq \bar{T}_a \implies \sigma \geq \bar{\sigma}$ $\quad$ $T_s \leq \bar{T}_s \implies \omega_n \geq \bar{\omega}_n$ $\quad$ $S \leq \bar{S} \implies \theta \leq \bar{\theta}$

Summarizing

Derive the unit step response of a second-order system with complex conjugate poles using Laplace domain analysis

Explain the influence of the natural frequency ω_n and damping ratio ξ on the transient response characteristics (e.g., overshoot, settling time, rise time)

Transform between pole representation and (ω_n, ξ) parametrization for underdamped second-order systems

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Self-assessment material

Question 1

In the context of second-order systems, what does ω_n represent in the standard notation?

Potential answers:

I: The damping coefficient
II: The natural frequency
III: The settling time constant
IV: The overshoot percentage
V: I do not know

Question 2

For a second-order system's step response, the settling time T_a is approximately:

Potential answers:

I: Inversely proportional to ω_n^2
II: Directly proportional to ξ^2
III: Inversely proportional to $\xi\omega_n$
IV: Independent of the damping ratio ξ
V: I do not know

Question 3

Which of the following correctly describes the relationship between the damping ratio ξ and the pole location?

Potential answers:

I: $\xi = \frac{\text{Im}[p]}{|p|}$

II: $\xi = -\frac{\text{Re}[p]}{|p|}$

III: $\xi = \frac{|p|}{\text{Re}[p]}$

IV: $\xi = \sqrt{1 - \left(\frac{\text{Im}[p]}{|p|}\right)^2}$

V: I do not know

Question 4

The overshoot S in a second-order system's step response:

Potential answers:

I: Increases with increasing damping ratio ξ
II: Decreases with increasing damping ratio ξ
III: Is independent of the damping ratio ξ
IV: Is always zero for $\xi > 0.5$
V: I do not know

Question 5

For a second-order system with complex conjugate poles, the step response contains:

Potential answers:

I: Pure exponential terms
II: A decaying sinusoidal term
III: A growing sinusoidal term
IV: A constant term only
V: I do not know

Recap of the module
"Analisi del transitorio per i sistemi del secondo ordine"

- per i sistemi del secondo ordine la risposta al gradino ha una espressione un po' piu' complicata, ed i vari indici hanno espressioni che e' meglio approssimare, per avere strumenti utilizzabili

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