state space representations

Contents map

developed content units	taxonomy levels
state of a system	u1, e1
separation principle	u1, e1

prerequisite content units	taxonomy levels
RR	u1, e1

Main ILO of sub-module "state space representations"

Define the meaning of "state space representation" in the context of linear and non-linear discrete time dynamical systems

Discussion: which information do you need to forecast accurately how long you may use your cellphone before its battery hits 0%?

Summarizing

these pieces of information contain all I need to forecast the future evolution of the battery level:

- current level of charge of the battery
- how much I will use the phone in the future
- how healthy the battery of my phone is
- which environmental factors may induce additional effects (too warm, too cold)

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$$\text{Time Remaining} = \frac{\text{Remaining Capacity}}{\text{Discharge Rate}}$$

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rewriting as a RR:

- $y[k] = Q[k] =$ remaining battery capacity at the discrete time kT (mAh)
- $u[k] = I[k] =$ current discharge rate at the discrete time kT (mA)

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$$\Longrightarrow \qquad y^+ = y - u$$

What is a state?

$$\begin{cases} x^+ & = -u \\ y & = x \end{cases}$$

"the current value of the state $x[k]$ contains all the information necessary to forecast the future evolution of the output $y[k]$ and of the state $x[k]$, assuming to know the future $u[k]$. I.e., to compute the future values $y[k+\kappa]$ and $x[k+\kappa]$ it is enough to know the current $x[k]$ and the current and future inputs $u[k:k+\kappa]$"

Question 1

In a spring-mass system, which of the following is a valid state variable?

Potential answers:

I: The temperature of the spring.
II: The displacement of the mass from its equilibrium position.
III: The color of the mass.
IV: The external force applied to the system.
V: I do not know.

Question 2

Which of the following pairs of variables can fully describe the state of a spring-mass system?

Potential answers:

I: The mass of the spring and the stiffness of the mass.
II: The external force and the displacement of the mass.
III: The displacement of the mass and the velocity of the mass.
IV: The acceleration of the mass and the color of the spring.
V: I do not know.

What do we mean with “modelling a state-space dynamical system”?

Defining

$$\begin{cases} \boldsymbol{x}^{+} & = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right) \\ \boldsymbol{y} & = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\theta}\right) \end{cases}$$

and

- the variables
  - $\boldsymbol{u}$ = the inputs
  - $\boldsymbol{d}$ = the disturbances
  - $\boldsymbol{x}$ = the states vector
  - $\boldsymbol{y}$ = the measured outputs
- the structure of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$
- the value of the parameters $\boldsymbol{\theta}$

Discrete time state space model - definition

Ingredients:

- the number of inputs, outputs and state variables must be finite
- the difference equations must be first order
- the separation principle *(the current value of the state contains all the information necessary to forecast the future evolution of the outputs and of the state)* shall be satisfied

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state space model = finite set of first-order difference equations that connect a finite set of inputs, outputs and state variables so that they satisfy the separation principle

State space representations - Notation

- $u_1, \ldots, u_m$ = inputs
- $x_1, \ldots, x_n$ = states
- $y_1, \ldots, y_p$ = outputs
- $d_1, \ldots, d_q$ = disturbances

State space representations - Notation

$$\begin{aligned} \boldsymbol{x}^{+} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

- $\boldsymbol{f}$ = state transition map
- $\boldsymbol{g}$ = output map

examples

Bank Account Balance

Scenario: modeling the balance of a bank account over time with monthly deposits and interest:

- $x[k]$ = account balance at month k
- $u[k]$ = monthly deposit/withdrawing at month k
- $r[k]$ = monthly interest rate *(may be time-varying!)*

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State-Space Model:

$$\begin{cases} x[k+1] = (1 + r[k]) \cdot x[k] + u[k] & \textit{(State equation)} \\ y[k] = x[k] & \textit{(Output equation)} \end{cases}$$

Particular case: exponential growth, scalar version

$$x^{+} = \alpha x + \beta u \qquad y = x$$

Exponential growth, matricial version

Generalization of all linear systems

$$\begin{cases} \boldsymbol{x}^{+} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{cases}$$

Lotka-Volterra

- y_{prey} := prey
- y_{pred} := predator

$$\begin{cases} y_{\text{prey}}^{+} & = \alpha y_{\text{prey}} - \beta y_{\text{prey}} y_{\text{pred}} \\ y_{\text{pred}}^{+} & = -\gamma y_{\text{pred}} + \delta y_{\text{prey}} y_{\text{pred}} \end{cases}$$

`./LotkaVolterraSimulator.ipynb`

Summarizing

Define the meaning of "state space representation" in the context of linear and non-linear discrete time dynamical systems

- recall the definition of state space model
- be sure to have interiorized the separation principle with some practical examples

Most important python code for this sub-module

Important library

https://python-control.readthedocs.io/en/0.10.1/conventions.html#
state-space-systems

Self-assessment material

Question 3

In the battery charge model $y^+ = y - u$, what does y represent?

Potential answers:

I: The discharge rate of the battery.
II: The remaining battery capacity.
III: The temperature of the battery.
IV: The external force applied to the battery.
V: I do not know.

Question 4

What does the separation principle in state-space models imply?

Potential answers:

I: The system must have an infinite number of states.
II: The system must be linear.
III: The current state contains all information needed to predict future outputs and states.
IV: The system must have no inputs or disturbances.
V: I do not know.

Question 5

In the bank account model $x[k+1] = (1 + r[k]) \cdot x[k] + u[k]$, what does $u[k]$ represent?

Potential answers:

I: The interest rate at month k.
II: The monthly deposit or withdrawal at month k.
III: The account balance at month k.
IV: The total interest earned at month k.
V: I do not know.

Question 6

In the Lotka-Volterra model, what does the term $\beta y_{\mathrm{prey}} y_{\mathrm{pred}}$ represent?

Potential answers:

I: The natural growth rate of the prey population.
II: The natural death rate of the predator population.
III: The interaction between prey and predator populations.
IV: The external disturbance affecting the system.
V: I do not know.

Question 7

In the exponential growth model $x^{+} = \alpha x + \beta u$, what does α represent?

Potential answers:

I: The growth rate of the system.
II: The input to the system.
III: The output of the system.
IV: The disturbance affecting the system.
V: I do not know.

Recap of sub-module “state space representations”

- a set of variables is a state vector if it satisfies for that model the separation principle, i.e., the current state vector “decouples” the past with the future
- state space models are finite, and first order vectorial models

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