Systems Laboratory, Spring 2025

Damiano Varagnolo –

Connections between eigendecompositions and free evolution in continuous time LTI state space systems

Contents map

developed content units	taxonomy levels
modal analysis	u1, e1

prerequisite content units	taxonomy levels
LTI ODE	u1, e1
state space system	u1, e1
eigenvalue	u1, e1
eigenspace	u1, e1
phase portrait	u1, e1

Main ILO of sub-module
"Connections between eigendecompositions and free evolution in continuous

Analyse the structure of the free evolution of the state variables by means of the eigendecomposition of the system matrix

Important initial remark

focus = LTI in state space and free evolution, meaning $u(t) = 0$, and thus

$$\begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} + Bu \\ y & = C\boldsymbol{x} \end{cases} \quad \mapsto \quad \begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} \\ y & = C\boldsymbol{x} \end{cases}$$

. . . and then an important disclaimer

$$\begin{cases} \dot{\boldsymbol{x}} & = A\boldsymbol{x} \\ y & = C\boldsymbol{x} \end{cases}$$

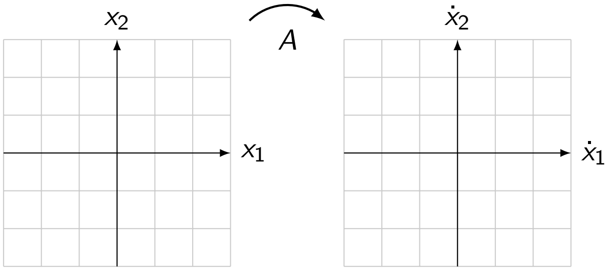
the module ignores what happens if A *is non-diagonalizable*

Roadmap

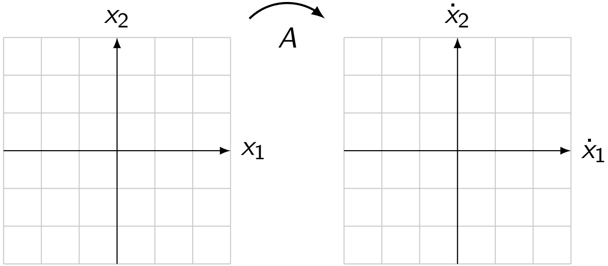
- set the focus just on $\boldsymbol{x}$, and not on $\boldsymbol{y}$
- get a graphical intuition of what $A\boldsymbol{x}$ means
- interpreting eigenspaces in the real of LTI continuous time systems
- adding the "superposition principle" ingredient to the mixture

What does Ax mean, graphically?

The physical meaning of the operation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

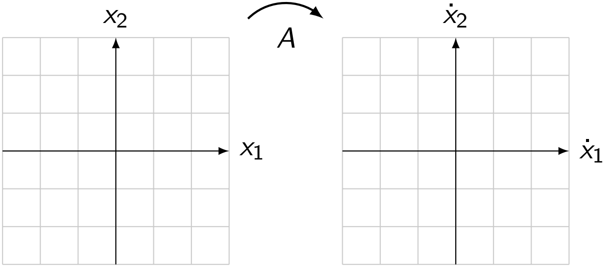


The physical meaning of the operation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$



$\Longrightarrow$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system.

The physical meaning of the operation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

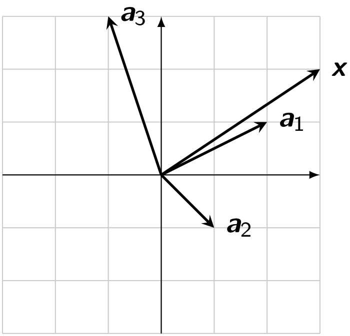


$\implies$ structure of A determines how the time derivative $\dot{\boldsymbol{x}}$ is, and how the time derivative is determines the stability and time-evolution properties of the system. E.g.,

$$\text{span}\,(A) = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \quad \implies \quad \text{if } x_1 \text{ grows then } x_2 \text{ diminishes, and viceversa}$$

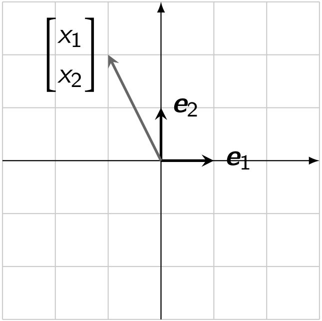
How may we represent vectors and matrices?

$$\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$



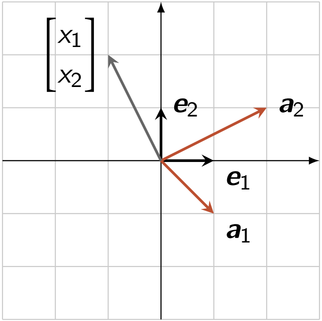
But what is a vector?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 = \boldsymbol{e}_1 x_1 + \boldsymbol{e}_2 x_2$$



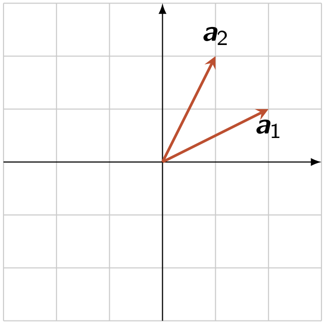
So, what is a matrix-vector product, geometrically?

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad A\boldsymbol{x} = ?$$



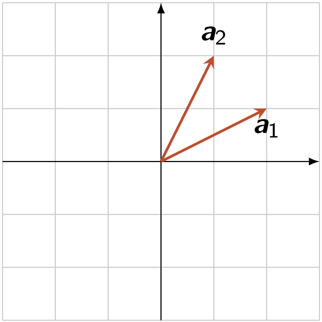
The effect of eigenspaces

Eigenvectors of a square matrix



are there some directions that get only stretched, i.e., that do not rotate?

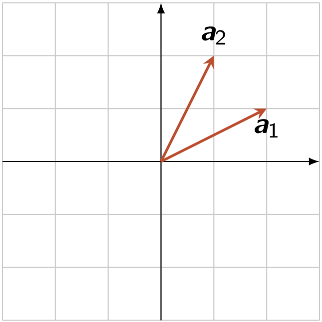
Eigenvectors of a square matrix



are there some directions that get only stretched, i.e., that do not rotate?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

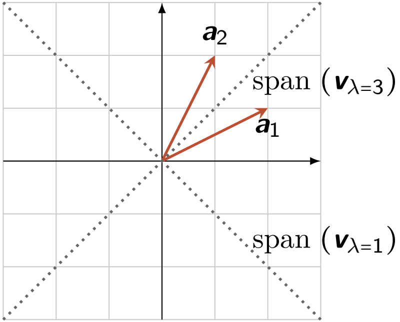
Eigenvectors of a square matrix



are there some directions that get only stretched, i.e., that do not rotate?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mapsto \quad \boldsymbol{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

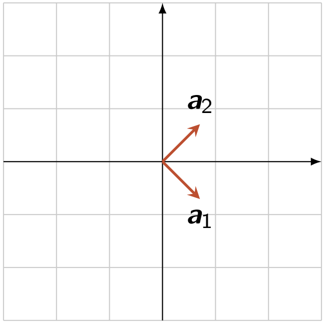
Eigenspaces = subspaces spanned by the eigenvectors-eigenvalues pairs



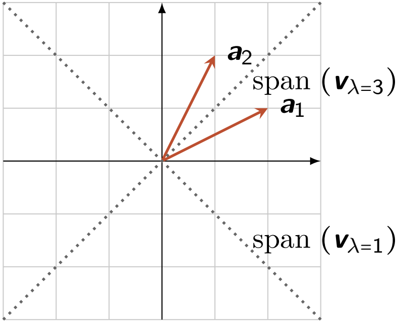
eigenspaces = subspaces spanned by the eigenvectors

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mapsto \quad \boldsymbol{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvectors: sometimes you may seem them from the transformation of the hypercube, sometimes you don't

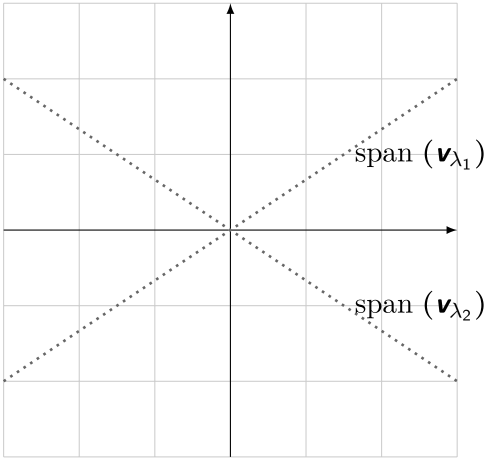


Why do we like eigenspaces?



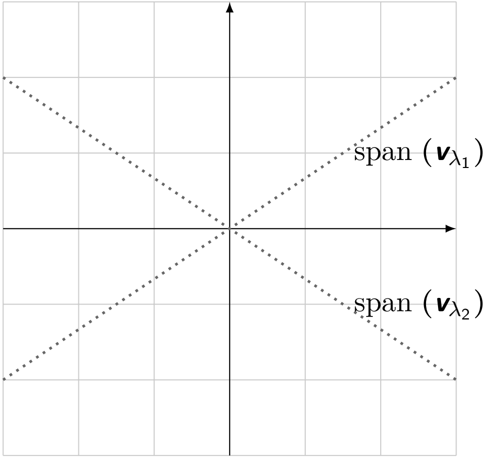
because $\dot{\boldsymbol{x}} = \lambda \boldsymbol{x} \implies$ "keep moving along that line"

Why do we like eigenspaces? Take 2



superposition principle $\implies$ one can caracterize the whole phase portrait

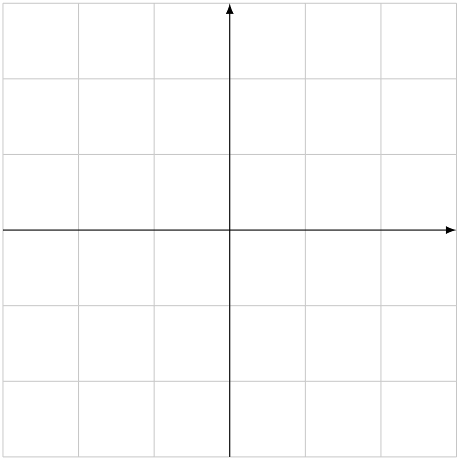
Why do we like eigenspaces? Take 3



the trajectory along each eigenspace is driven by a first order differential equation

$\implies$ if $\boldsymbol{x}_0 \in \mathrm{span}(\boldsymbol{v}_\lambda)$, then $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{x}_0$

Examples



How do we compute eigenvalues and eigenvectors numerically?

```
eigenvalues, eigenvectors = numpy.linalg.eig(A)
```

Summarizing

Analyse the structure of the free evolution of the state variables by means of the eigendecomposition of the system matrix

- find the eigenspaces and the eigenvalues
- depending on the values of the eigenvalues, understand how the trajectories along the eigenspaces look like
- depending on the relative angle among the eigenspaces, infer the phase portrait
- if the system matrix is not diagonalizable, then this concept complicates due to the presence of generalized eigenspaces (not in this module)

Most important python code for this sub-module

Linear algebra in general

`https://numpy.org/doc/2.1/reference/routines.linalg.html`

Self-assessment material

Question 1

What does a positive eigenvalue imply about the system's behavior along its corresponding eigenspace?

Potential answers:

I: The state grows exponentially along that eigenspace.
II: The state decays exponentially along that eigenspace.
III: The state oscillates along that eigenspace.
IV: The state remains constant along that eigenspace.
V: I do not know.

Question 2

In the context of free evolution of a linear time-invariant (LTI) system, what does the equation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ represent?

Potential answers:

I: The evolution of the system's output over time.

II: The evolution of the state variables over time, influenced by the system matrix A.

III: The relationship between input and output signals in the system.

IV: The response of the system to external inputs.

V: I do not know

Question 3

Why is it useful to consider the eigendecomposition of the system matrix A in analyzing the free evolution of state variables?

Potential answers:

I: It simplifies calculating the system's forced response.

II: It directly determines the output y of the system.

III: It helps identify invariant directions (eigenvectors) and growth/decay rates (eigenvalues) that govern the system's behavior over time.

IV: It only affects the graphical representation, not the actual system behavior.

V: I do not know

Question 4

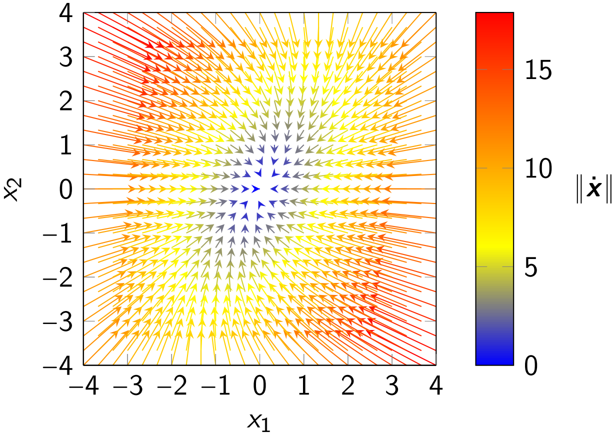
In a graphical representation, what does the matrix-vector product $A\boldsymbol{x}$ illustrate in the context of system dynamics?

Potential answers:

I: The projection of the state vector onto the output space.
II: The response of the system to a unit impulse.
III: Where the trajectory of the system is going, starting from $\boldsymbol{x}$.
IV: The change in the input signal over time.
V: I do not know

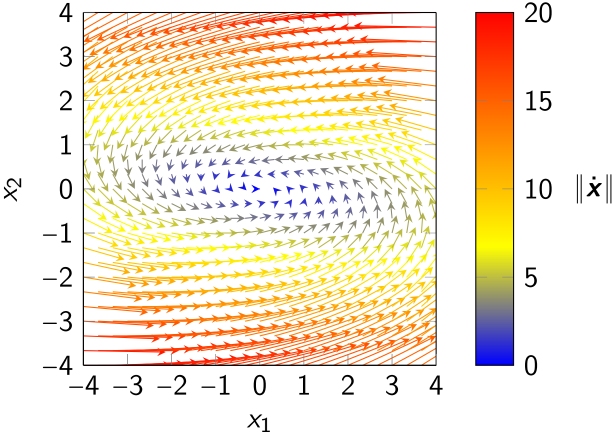
Question 5

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



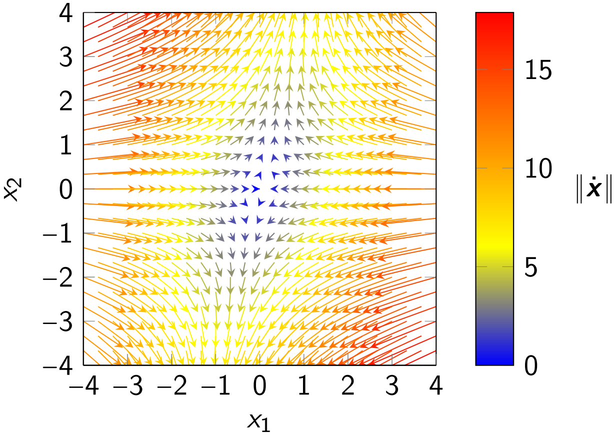
Question 6

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



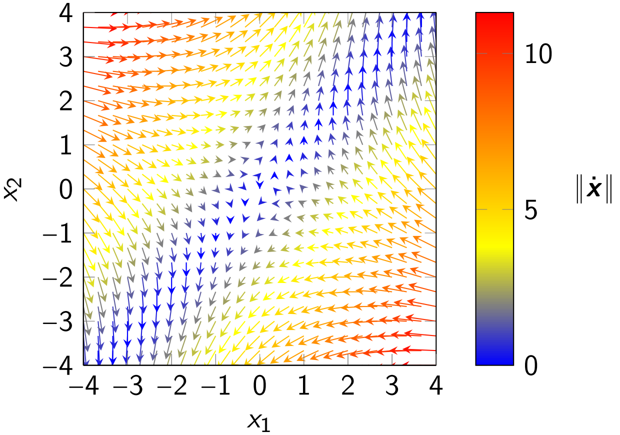
Question 7

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



Question 8

Which eigenvalues and eigenspaces would you say characterize the system matrix A, looking just at this phase portrait?



Recap of sub-module
“Connections between eigendecompositions and free evolution in continuous

- the eigenvalues of the system matrix A give the growth / decay rates of the modes $e^{\alpha t}$ of the free evolution of the system
- along eigenspaces, the trajectory of the free evolution is “simple”, i.e., aligned with that eigenspace
- the kernel of the system matrix gives us the equilibria corresponding to $u = 0$

?