computing free evolutions and forced responses of LTI systems

Contents map

developed content units	**taxonomy levels**
free evolution	u1, e1
forced response	u1, e1

prerequisite content units	**taxonomy levels**
LTI ODE	u1, e1
convolution	u1, e1
partial fraction decomposition	u1, e1

Main ILO of sub-module "computing free evolutions and forced responses of LTI systems"

Compute free evolutions and forced responses of LTI systems
using Laplace-based formulas (but only as procedural tools)

Disclaimer

the formulas introduced in this module shall be taken as “ex machina”

Focus in this module = on ARMA models

$$y^{(n)} = a_{n-1} y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$$

with $^{(i)}$ meaning the i-th time derivative.

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$$y^{(n)} = a_{n-1} y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$$

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Laplace transforms - links for who would like to get more info about them

Laplace transforms = extension of Fourier transforms; interesting material:

- `https://www.youtube.com/watch?v=r6sGWTCMz2k` (Fourier series)
- `https://www.youtube.com/watch?v=spUNpyF58BY` (Fourier transforms)
- `https://www.youtube.com/watch?v=nmgFG7PUHfo` (on the historical importance of Fast Fourier Transforms)
- `https://www.youtube.com/watch?v=7UvtU75NXTg` (Laplace Transforms, in math)
- `https://www.youtube.com/watch?v=n2y7n6jw5d0` (Laplace Transforms, graphically)

Main usefulness: convolution in time transforms into multiplication in Laplace-domain, and viceversa

$$\begin{cases} H(s) = \mathcal{L}\{h(t)\} \\ U(s) = \mathcal{L}\{u(t)\} \end{cases} \quad \Longrightarrow \quad \mathcal{L}\{h * u(t)\} = H(s)U(s)$$

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$$\begin{cases} H(s) = \mathcal{L}\{h(t)\} \\ U(s) = \mathcal{L}\{u(t)\} \end{cases} \implies \mathcal{L}\{h * u(t)\} = H(s)U(s)$$

Noticeable name: *transfer function* $(= H(s) = \mathcal{L}\{\text{impulse response}\})$

An intuitive explanation of the usefulness of the Laplace transform in automatic control

initial value problem

$$\dot{x}(t) + x(t) = 0, \quad x(0) = 2$$

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initial value problem

$$\dot{x}(t) + x(t) = 0, \quad x(0) = 2$$

$\mathcal{L}$

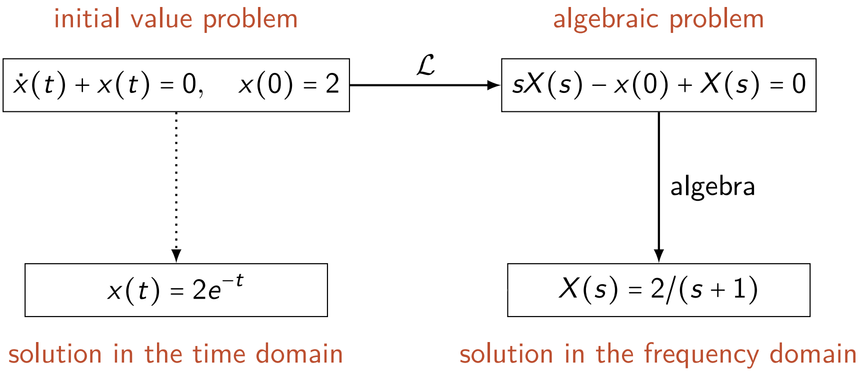
algebraic problem

$$sX(s) - x(0) + X(s) = 0$$

$$x(t) = 2e^{-t}$$

solution in the time domain

An intuitive explanation of the usefulness of the Laplace transform in automatic control



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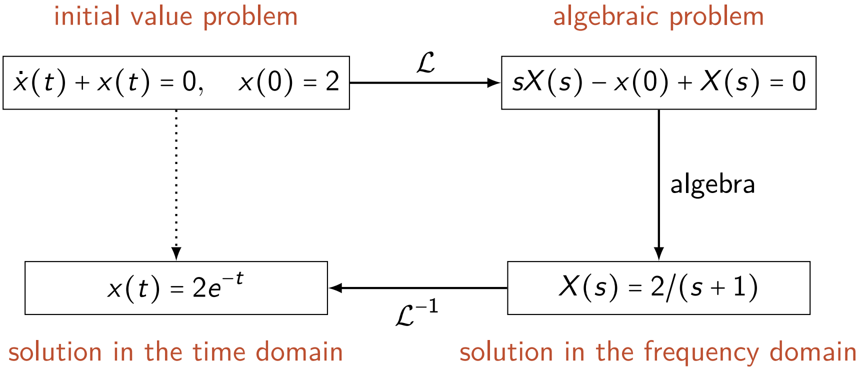

First set of formulas to memorize: Laplace-transforming derivatives

(these will be motivated in other courses)

$$\mathcal{L}\left\{\dot{x}\right\} = sX(s) - x(0)$$

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$$\mathcal{L}\left\{x^m\right\} = \ldots$$

Example: spring mass system

$$\ddot{y} = -\frac{f}{m}\dot{y} - \frac{k}{m}y + u$$

$$\Downarrow$$

$$s^2 Y(s) - s y_0 - \dot{y}_0 = -\frac{f}{m}\Big(sY(s) - y_0\Big) - \frac{k}{m}Y(s) + U(s)$$

$$\Downarrow$$

$$s^2 Y(s) + \frac{f}{m}sY(s) + \frac{k}{m}Y(s) = +s y_0 + \dot{y}_0 + \frac{f}{m}y_0 + U(s)$$

$$\Downarrow$$

$$Y(s) = \frac{y_0\left(\frac{f}{m} + s\right) + \dot{y}_0}{s^2 + \frac{f}{m}s + \frac{k}{m}} + \frac{1}{s^2 + \frac{f}{m}s + \frac{k}{m}}U(s)$$

And what shall we do once we get this?

$$\text{generalizing the previous slide: } Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} U(s)$$

with

- $\dfrac{M(s)}{A(s)}$ = Laplace transform of the free evolution
- $\dfrac{B(s)}{A(s)} U(s)$ = Laplace transform of the forced response

$\Longrightarrow$ we shall anti-transform; how?

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- $\frac{M(s)}{A(s)}$ = Laplace transform of the free evolution
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$\Longrightarrow$ we shall anti-transform; how? Main 2 cases:

- either $U(s) = \frac{\text{polynomial in } s}{\text{polynomial in } s}$
- or $U(s)$ = something else

Question 1

Is the Laplace transform of the signal

$$h(t) = \begin{cases} \dfrac{1}{t+1} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

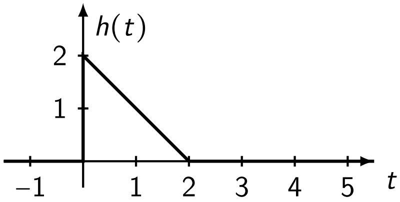
a rational Laplace transform?

Potential answers:

I: yes
II: no
III: it depends
IV: I don't know

Question 2

Is the Laplace transform of the signal $h(t)$ below a rational Laplace transform?



Potential answers:

I: yes

II: no

III: it depends

IV: I don't know

first case: rational $U(s)$

How to do if $U(s) = \dfrac{\text{polynomial in } s}{\text{polynomial in } s}$

$$Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} U(s) \quad \mapsto \quad Y(s) = \frac{M(s)}{A(s)} + \frac{C(s)}{D(s)}$$

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write each of the two parts of the signal as

$$\frac{N(s)}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3) \cdots}$$

Next step: partial fraction decomposition

- **case single poles:** if $\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

$$\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots} = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \frac{\alpha_3}{s-\lambda_3} + \cdots \qquad (1)$$

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- **case repeated poles:** if some poles are repeated, then there exist $\alpha_{1,1}, \ldots, \alpha_{1,n1}, \alpha_{2,1}, \ldots, \alpha_{2,n2}, \ldots,$ s.t.

$$\frac{N(s)}{(s-\lambda_1)^{n1}(s-\lambda_2)^{n2}\ldots} = \frac{\alpha_{1,1}}{s-\lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s-\lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s-\lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s-\lambda_2)^{n2}} + \ldots \tag{2}$$

Next step: partial fraction decomposition

- **case single poles:** if $\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\cdots}$ is s.t. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots$ then there exist $\alpha_1, \alpha_2, \alpha_3, \ldots$ s.t.

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- **case repeated poles:** if some poles are repeated, then there exist $\alpha_{1,1}, \ldots, \alpha_{1,n1}, \alpha_{2,1}, \ldots, \alpha_{2,n2}, \ldots,$ s.t.

$$\frac{N(s)}{(s-\lambda_1)^{n1}(s-\lambda_2)^{n2}\ldots} = \frac{\alpha_{1,1}}{s-\lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s-\lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s-\lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s-\lambda_2)^{n2}} + \ldots \tag{2}$$

"But how do I compute α_1, α_2, etc.?" $\mapsto$

`en.wikipedia.org/wiki/Partial_fraction_decomposition`

(tip: start from `en.wikipedia.org/wiki/Heaviside_cover-up_method`)

Anti-transforming in the rational $U(s)$ case

$$\text{if } Y(s) = \frac{\alpha_{1,1}}{s - \lambda_1} + \ldots + \frac{\alpha_{1,n1}}{(s - \lambda_1)^{n1}} + \frac{\alpha_{2,1}}{s - \lambda_2} + \ldots + \frac{\alpha_{2,n2}}{(s - \lambda_2)^{n2}} + \ldots \text{ then use}$$

$$\mathcal{L}\left\{t^n e^{\lambda t}\right\} = \frac{n!}{(s - \lambda)^{n+1}} \quad \leftrightarrow \quad \mathcal{L}^{-1}\left\{\frac{n!}{(s - \lambda)^{n+1}}\right\} = t^n e^{\lambda t}$$

Numerical Example: Inverse Laplace Transform of a Rational Function

$$Y(s) = \frac{3}{s-2} + \frac{4}{(s-2)^2} + \frac{5}{s+1}$$

goal = compute the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Step 1: Identify the terms

$$Y(s) = \frac{3}{s-2} + \frac{4}{(s-2)^2} + \frac{5}{s+1}$$

Here:

- $\lambda_1 = 2$, with coefficients $\alpha_{1,1} = 3$ and $\alpha_{1,2} = 4$
- $\lambda_2 = -1$, with coefficient $\alpha_{2,1} = 5$

Step 2: Apply the inverse Laplace transform formula

by means of

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-\lambda)^{n+1}}\right\} = t^n e^{\lambda t}$$

we compute the inverse Laplace transform of each term:

- $\mathcal{L}^{-1}\left\{\frac{3}{s-2}\right\} = 3e^{2t}$
- $\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2}\right\} = 4te^{2t}$
- $\mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} = 5e^{-t}$

Step 3: Combine the results

then we have that the inverse Laplace transform $y(t)$ is the sum of the individual transforms, i.e.,

$$y(t) = 3e^{2t} + 4te^{2t} + 5e^{-t}$$

Another Example: Inverse Laplace Transform with Complex Conjugate Terms

let

$$Y(s) = \frac{2s + 3}{s^2 + 2s + 5}$$

and the goal to be to compute the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$

Step 1: Factor the denominator

note: $s^2 + 2s + 5$ has complex conjugate roots, indeed

$$s^2 + 2s + 5 = (s+1)^2 + 4$$

and thus

$$Y(s) = \frac{2s+3}{(s+1)^2 + 4}$$

Step 2: Express in terms of standard forms

rewrite $Y(s)$ to match the standard forms for inverse Laplace transforms involving complex conjugates, i.e.,

$$Y(s) = \frac{2(s+1)+1}{(s+1)^2+4} = 2 \cdot \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}.$$

Step 3: Apply the inverse Laplace transform formula

since

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos(bt),$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = e^{-at}\sin(bt),$$

we have, for the various terms:

- $\mathcal{L}^{-1}\left\{2\cdot\frac{s+1}{(s+1)^2+4}\right\} = 2e^{-t}\cos(2t)$
- $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}e^{-t}\sin(2t)$

Step 4: Combine the results

$$y(t) = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$$

Extremely important result

a LTI in free evolution behaves as a combination
of terms $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc. for a set of different
λ's and powers of t, called the *modes* of the system

Extremely important result

a LTI in free evolution behaves as a combination of terms $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc. for a set of different λ's and powers of t, called the *modes* of the system

Discussion: assuming that we have two modes, $e^{-0.3t}$ and $e^{-1.6t}$, so that

$$y(t) = \alpha_1 e^{-0.3t} + \alpha_2 e^{-1.6t}.$$

What determines α_1 and α_2?

second case: irrational $U(s)$

In this case we cannot use partial fractions decompositions as before

$$\text{from } Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} U(s) \text{ we follow the algorithm}$$

- find $y_{\text{free}}(t)$ from PFDs of $\dfrac{M(s)}{A(s)}$ as before
- find the impulse response $h(t)$ from PFDs of $\dfrac{B(s)}{A(s)}$ as before
- find $y_{\text{forced}}(t)$ as $h * u(t)$

Summarizing

Compute free evolutions and forced responses of LTI systems
using Laplace-based formulas (but only as procedural tools)

- Laplace the ARMA
- if $u(t)$ admits a rational $U(s)$ then write $Y(s) = \frac{\text{polynomial}}{\text{polynomial}}$, do PFD, and do inverse-Laplaces
- if $u(t)$ does not admit a rational $U(s)$, do similarly as before but do PFD only for the free evolution and impulse response, and find the forced response by means of convolution

Most important python code for this sub-module

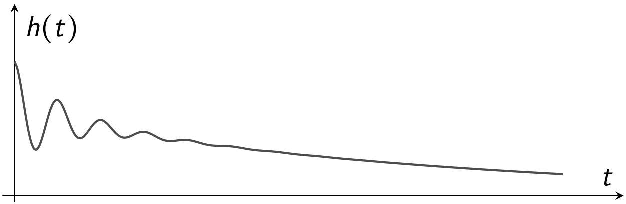
Two essential libraries

- `https://python-control.readthedocs.io/en/0.10.1/generated/control.modal_form.html`
- `https://docs.sympy.org/latest/modules/physics/control/lti.html`

Self-assessment material

Question 3

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: first order

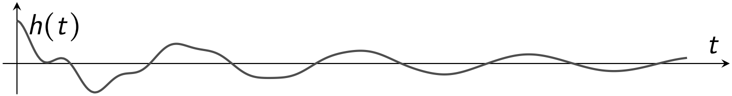
II: second order

III: at least third order

IV: I do not know

Question 4

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?

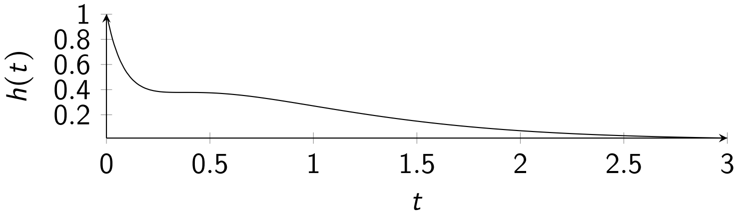


Potential answers:

I: first order
II: second order
III: third order
IV: at least fourth order
V: I do not know

Question 5

Which type of LTI system may produce the impulse response $h(t)$ below?



Potential answers:

I: first order

II: second order

III: at least third order

IV: I do not know

Question 6

What is the primary purpose of using Laplace transforms in solving LTI systems?

Potential answers:

I: To convert differential equations into algebraic equations for easier solving.
II: To transform convolution in the time domain into multiplication in the Laplace domain.
III: To directly compute the eigenvalues of the system matrix.
IV: To eliminate the need for initial conditions in solving differential equations.
V: I do not know.

Question 7

What is the correct form of the inverse Laplace transform of $\dfrac{1}{(s-\lambda)^2}$?

Potential answers:

I: $e^{\lambda t}$

II: $te^{\lambda t}$

III: $te^{\lambda t}$

IV: $\frac{1}{2}t^2 e^{\lambda t}$

V: I do not know.

Question 8

What is the inverse Laplace transform of $\frac{s+1}{(s+1)^2+4}$?

Potential answers:

I: $e^{-t}\sin(2t)$

II: $e^{-t}\cos(2t)$

III: $e^{-t}\cos(t)$

IV: $e^{-t}\sin(t)$

V: I do not know.

Question 9

In the ARMA model $y^{(n)} = a_{n-1}y^{(n-1)} + \ldots + a_0 y + b_m u^{(m)} + \ldots + b_0 u$, why is the leading coefficient of $y^{(n)}$ typically set to 1?

Potential answers:

I: To ensure the system is stable.
II: To simplify the computation of eigenvalues.
III: To reduce the number of parameters and work with monic polynomials.
IV: To make the system linear time-invariant.
V: I do not know.

Question 10

What determines the coefficients α_1 and α_2 in the free evolution response $y(t) = \alpha_1 e^{-0.3t} + \alpha_2 e^{-1.6t}$?

Potential answers:

I: The eigenvalues of the system matrix.
II: The input signal $u(t)$.
III: The initial conditions of the system.
IV: The poles of the transfer function.
V: I do not know.

Recap of sub-module “computing free evolutions and forced responses of LTI systems”

- finding such signals require knowing a couple of formulas by heart
- partial fraction decomposition is king here, one needs to know how to do that

?